The mirror does not lie

Endogenous fiscal limits for Slovakia

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Abstract

We study the interactions among fiscal policy, fiscal limits and the associated sovereign risk premium. The fiscal limit distribution, which measures the ability of the government to service its debt, arises endogenously from dynamic Laffer curves. We assume a feedback loop between the fiscal limit distribution and the risk premium and determine them simultaneously using and efficient iterative scheme. A nonlinear relationship between the sovereign risk premium and the level of government debt then emerges in equilibrium. The model is calibrated to Slovak data assuming steeply growing age-related transfers and volatile business cycle. We study the impact of various model parameters on the conditional (state-dependent) and unconditional distributions of the fiscal limit. Fiscal limit distributions obtained via Markov–Chain–Monte–Carlo regime switching algorithm depend on the rate of growth of government transfers, the degree of countercyclicality of policy, and the distribution of the underlying economic conditions. We find that both distributions are considerably more heavy-tailed compared with those usually obtained in the literature for advanced economies, and are very sensitive to the size and rate of growth of transfers, the business cycle phase and the fiscal policy credibility. The main policy message is that the Maastricht debt limit of 60 percent of GDP is not safe enough for Slovakia. Furthermore, credible reforms reining in age-related spending and thus stabilising public finance in the long-run, should be a priority.

Keywords: Simulation Methods and Modelling, Fiscal Policy, Government Expenditures, Debt Management and Sovereign Debt.

JEL Classification: C15, C63, E62, H5, H63.

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Introduction

The theoretical analysis of fiscal policy in advanced economies has traditionally abstracted from sovereign default risk. Although for many years, sovereign defaults have been studied mainly in the context of emerging markets, nowadays due to the recent financial crisis, and the resulting rise of public debt in developed countries, the importance of the debt sustainability and default risk took center stage, especially in the euro area. Moreover, age–related expenditures represent another source concern about the long-term sustainability of public finances. Therefore, it is essential to understand the interaction between sovereign default risk and fiscal policy and, furthermore, to discuss what kind of fiscal policies can contain the default risk. The Council for Budget Responsibility (CBR) evaluates the long-term sustainability of public finances. The analysis of the relationship between the fiscal policy and default risk represent a substantial part of the risk assessment building block in the CBR's toolkit[1].

In this paper we present one possible approach that enables us to study the relationship between the fiscal policy, and the resulting fiscal limit and the risk premium. Following Bi (2012) and Bi and Leeper (2013), we construct a simple real business cycle nonlinear model that allows us to describe, how the *fiscal limit*, the maximum level of debt that the government is able to service, depends on macroeconomic fundamentals[2]. We also calculate the risk premium that emerge from agents' intertemporal choice taking into account the fact that the government might default on its outstanding liabilities. Moreover, by studying the state-independent unconditional distribution of the fiscal limits, we are able to analyse the long-run impacts of various fiscal policy reforms.

Our research extends the previous works (Mucka (2015), Bi (2012) and Bi and Leeper (2013)) in one crucial aspect. Concretely, by introducing the feedback from financial markets to fiscal authority we remove the major drawback of the former studies – the independence of fiscal limit distributions on the risky interest rate charged by investors. Therefore in this paper we enhance our previous study (Mucka (2015)) and formulate the coupled problem of finding the fiscal limit simultaneously with the corresponding default risk premium using an efficient iterative scheme. This change has a crucial impact on the distribution of the fiscal limit and so the government expectations about the long-term fiscal sustainability. In former studies a single deterministic exogenous rate was used to discount expected future primary surpluses. However, employing bond-pricing approach in our study it turns out that the state-dependant discount rate becomes stochastic and purely endogenous as it reflects not only the response from financial markets but also the debt level and fiscal policy sustainability.[3].

We emphasize that our study enrich Bi (2012) and Bi and Leeper (2013) in two additional ways to make it more relevant in the context of Slovakia[4]. The underlying growth in government trans-

[1]Our study is the first step in the process of the safe debt level analysis. The conceptual framework of the CBR fiscal risk assessment is explained in detail in Odor and Jurasekova-Kucserova (2014).

[2]Fiscal limit is defined as the sum of the discounted maximum fiscal primary surplus in all future periods

[3]Moreover, from the technical perspective, we increase the accuracy of the numerical solution by employing more proper grid specification with non-equidistant nodes, combining various numerical quadrature methods (Gauss-Legendre, Simpson's 3/8 rule) and interpolating using cubic splines.

[4]These features were introduced in our previous paper (Mucka (2015)). Furthermore, our recent work assumed the

fers was calibrated to reflect the ageing population of Slovakia. Moreover, we allowed transfers to follow a regime–switching process that better reflects the political cycle in Slovakia. Finally, we drew total factor productivity – the only source of business cycle fluctuations in the model – from a heavy-tailed distribution that approximated empirical cyclical conditions in Slovakia very well.

In this context, we find that the feedback from financial markets to fiscal authority has an essential impact on the distribution of the fiscal limits. and the rate of growth of transfers in the economy, the credibility of the fiscal policy, the degree to which policies respond to the economic cycle, and the distribution of cyclical conditions in the economy all affect the distribution of the fiscal limit significantly and in both the short- and long-run. We find that the distribution of the debt limit is considerably heavy-tailed, and an adverse combination of conditions and policies could generate high probabilities of default even at debt levels generally considered to be safe. Hence, our main policy conclusion is that the debt limit enshrined in the Stability and Growth Pact of 60 percent of GDP may not be ”safe” for Slovakia.

Although in normal times the Maastricht debt limit is associated with only a 9 percent chance of default, this probability rises sharply to 39 percent when the country faces a significant drop in the productivity level. A similar effect (29 percent increase) is observed in case of essentially higher initial level of transfers. Furthermore, running a bad fiscal policy in bad times, in other words allowing the age-related expenditures to rise faster relative to the no-policy-change scenario implies even a 50 percent chance that country would default on its liabilities. These conclusions are even amplified when the long-run impacts of the policy reforms are studied. Therefore, governments would be well–advised to keep debt levels at a significantly lower level. To do so, it appears crucial to control the long-term growth of transfers. Hence, credible reforms to age-related spending should be a priority.

The rest of the paper is organized as follows. The first part of the paper offers a brief overview of the relevant literature. The second section introduces the modelling framework, the concept of coupled problem and describe the problem solution. The third part calibrates the model to Slovak data. The forth section contains our quantitative exercise with regards the distribution of the fiscal limit and the risk premium. The last section concludes and presents several avenues for further research.

existence of automatic stabilizers in the fiscal policy as the response of the economic cycle to both transfers and government spending was incorporated.

1 Related literature

For many years, literature on sovereign defaults have been focused mostly on emerging markets. However, recent recession and financial crisis caused raising importance of the fiscal sustainability for policy makers and financial markets in developed economies, especially euro area and led to increased number of studies scrutinizing the fiscal sustainability issues. D'Erasmo et al. (2016) critically review traditional methods and recommend three alternative approaches to find out what is a sustainable public debt.

The first one is a method based on the estimation of fiscal reaction functions (Bohn (1998) and Bohn (2008)). Ghosh et al. (2011) use this approach to calculate debt limits and measure fiscal space in advanced countries. They estimate the responses of primary fiscal surplus to debt levels and compute debt limits and fiscal space based on historical fiscal rules. As their calculations are backward-looking, grounded in past policies that are assumed to be immutable, reforms in the fiscal policy currently implemented in European countries invalidate their estimates of the debt limits and fiscal space.

The another approach recommended by D'Erasmo et al. (2016) assumes that governments cannot commit to repay debt and can thus optimally decide to default. Models of strategic default were pioneered by the seminal contribution of Eaton and Gersovitz (1981). These models feature endogenous sovereign spreads, a wealth maximizing government and endogenous borrowing policies. As suggested by Bi and Leeper (2013), sovereign default risk helps standard RBC models reproduce key business cycle facts in emerging economies, particularly countercylical interest rates and net exports, and volatile consumption. By modelling default as an optimal response to exogenous shocks, however, the strategic default literature is largely silent about the policy behaviour that led the country into a sovereign debt crisis in the first place and also about the policy reforms that might resolve the crisis.
Aguiar and Gopinath (2006) developed a quantitative model of debt and default in a small open economy, where defaults occur in equilibrium. In their study, inspired by Eaton and Gersovitz (1981) sovereign default arises as an optimal response to productivity shocks. Their model was able to match several emerging market empirical regularities: counter-cyclicality of interest rates and net exports and positive correlations between interest rates and current accounts. However, they used only one-period debt and simulated debt and spread levels were low compared to their empirical counterparts. With the aim of building more realistic models, several additional features were introduced into the basic framework. Arellano (2008) added non-linear income cost of defaulting, while Hatchondo and Martinez (2009) introduced the possibility of issuing long-term debt. Chatterjee and Eyigungor (2012), Hatchondo et al. (2016) and Aguiar et al. (2016) all emphasize that the presence of long-term debt in the model is crucial in order to match empirical regularities in the data.
Once long-term debt is embedded into the model, debt dilution arises almost automatically, because existing sovereign debt contracts do not address this externality. In other words, if one wants to replicate realistic sovereign debt and sovereign default premiums, the problem of debt dilution cannot be avoided. Juessen et al. (2009) studies government repayment capacity using Laffer curve approach and emphasize the productivity shock importance. However, with their assumption on constant tax rate, current debt exceeds the debt limit, default occurs in an

amount necessary for equilibrium. They show that sovereign default risk premia turn out to emerge at either very high debt to output ratios, or if the variance of productivity shocks is large.

We emphasize that the strategic default literature remains very parsimonious when describing the policy behaviour that led the country into a sovereign debt crisis but also about the appropriate fiscal policy actions that might lead resolution of the crisis. These issues are widely discussed in studies employing a structural model based on a dynamic general equilibrium framework with a fully specified fiscal sector. They represent the third approach recommended by D'Erasmo et al. (2016). A pioneering work in this field is a concept of a debt limit, first used by Bi (2012), Bi and Leeper (2010), Bi and Traum (2012) and Bi and Leeper (2013) in which they answer: given the economic environment, what is the distribution of government debt that can be supported?. The fiscal limit arises endogenously from the peak of the Laffer curve, distribution of economic shocks and expectations about future policies. By mapping economic environment, especially fiscal policy regimes, into fiscal limit distributions and sovereign bond prices, they provide a tool to examine the efficacy of fiscal reforms pursued by countries that are under sovereign risk pressures. Bi (2012) and Bi and Leeper (2013) illustrate the usage of this approach by estimating the conditional[5] fiscal limit distribution for Greece facing the recent debt crisis. Furthermore, Bi and Leeper (2010) study the impact of long-run fiscal reforms implemented in Sweden by studying the corresponding unconditional fiscal limit distribution. They show how changes in fiscal behaviour shift fiscal limit distribution and affect sensitivity of risk premia to debt levels and emphasize that both the nature and the credibility of proposed reforms matter for their likely success in reducing sovereign risk. Hence even when fundamentals are poor credible shifts to a stabilizing regime can lead to risk premium reduction.

[5]Referring to Bi and Leeper (2013) a conditional distribution reflects the notion that bondholders expectations of repayment depend on the current state of the economy, including shock realizations and the policy regime.

2 Modelling framework

The original model of (Bi (2012) and Bi and Leeper (2013)) considers a closed economy in which the government finances lump-sum transfers to homogeneous households and an exogenous level of purchases, produced along with consumption goods using a simple linear production function, by collecting distorting taxes levied on labour and issuing non-state- contingent debt. The government raises the time-varying tax rate when the debt level grows. Laffer curves arise endogenously from distorting taxes - if the tax rate is on the slippery side of the Laffer curve, then the government is unable to raise more tax revenue through higher tax rates. The lump-sum transfers follow a Markov regime-switching process, with one regime being stationary while the other explosive. If the government stays in the explosive-transfer regime for too long, the debt can rise to such a level that tax rate may eventually reach the peak of Laffer curve and the government will be unable to repay its debt in full amount. Even if the current tax rate is not there yet, a positive probability of eventually hitting the peak of Laffer curve in the future can prompt forward-looking households to demand a higher default risk premium on sovereign debt today.

Therefore, the concept of the *fiscal limit* is introduced, as the maximum level of debt that the government is able to service, which is defined as the sum of the discounted maximum fiscal primary surplus in all future periods. It is the point at which, for economic or political reasons, the government can no longer adjust taxes and spending (government consumption and transfers to households) to stabilize debt. An estimate of the tax rate at which the peak of the Laffer curve is also obtained. Given the persistence of exogenous disturbances, the conditional distribution of the fiscal limit depends on the current state of the economy (productivity level, regime of transfers, level of government purchase) and on random disturbances hitting the economy in the future. The fiscal limit is state-dependent and has a stochastic distribution. However, for some analysis, particularly of long-run fiscal reforms, the unconditional, state-independent fiscal limit distribution is more appropriate.

To estimate country default risk premium one must solve the nonlinear model which uses the already calculated distribution of the fiscal limit. Even this simple model generates non-linearities that play a critical role in pricing sovereign debt. Due to the high non-linearities and the discontinuity one cannot employ log–linearisation to solve the model. Instead, the procedure is as follows. At each period, an effective fiscal limit is drawn from the state-dependent fiscal limit distribution. If the level of government debt hits the effective fiscal limit, then the government reneges on a fraction of its debt and the realized default rate follows an empirical distribution that is computed from historical data. Otherwise, the debt is repaid in full. Using the state-dependent distributions of fiscal limits and the empirical distribution of default rates, households can decide the quantity of government debts that they are willing to purchase and the price at which they are willing to pay. Furthermore, households make a decision about their level of consumption and labour supply, pay tax from their labour income and receive lump-sum transfers at level determined by the fiscal authority. The government collects tax revenues levied with rate reflecting current post–default debt from which it finances unproductive[6] spending (government purchase) and transfers following the Markov–switching regime. The accumulated debt is priced by the forward–looking households considering the current primary surplus and their expectations of the next–period default rate and future changes in consumption. Then the default risk premium arises as the difference between this risky interest rate and the rate calculated under assumption of no default.

In our closed-economy RBC model with fixed capital a representative firm employs a simple linear production technology to produce homogeneous final goods. Therefore, the output depending on the level of productivity a_t and household labour supply h_t is purchased by government g_t or consumed by households as expressed in the following aggregate resource constraint:

$$a_t h_t = y_t = c_t + g_t \,. \tag{1}$$

We assume that the deviation of the productivity from its steady state follows an autoregressive process

$$a_t = [a_{t-1}]^{\rho_a} [a]^{1-\rho_a} \exp\{\varepsilon_t^a\} \,. \tag{2}$$

with the shock process $\varepsilon_t^a \sim \mathcal{N}(0, \sigma_a^2)$[7].

Under the assumption of complete asset markets, government finances their purchase g_t and lump-sum transfers to households z_t by issuing one–period bonds b_t with price q_t and collecting a levied tax τ_t on labour income. For each unit of the bond purchased in the beginning of the period, the government promises to pay the household one unit of consumption in the next period. The bond contract is not enforceable since at time t a partial default of fraction Δ_t on government liability issued in the beginning of that time period b_{t-1} is possible.
Therefore, denoting the post–default debt liability b_t^d,

$$b_t^d \equiv (1 - \Delta_t) b_{t-1} \,, \tag{3}$$

the government budget satisfies:

$$q_t b_t - b_t^d = z_t + g_t - a_t h_t \tau_t \,. \tag{4}$$

The default scheme at each period depends on the effective fiscal limit b_t^* drawn from an endogenously determined conditional distribution $\mathscr{B}^*$. If government liability at the beginning of period t does not attain the effective fiscal limit, then no default occurs since it repays its debt in full amount. Otherwise, a partial default takes place and the stochastic default rate follows an empirical distribution Ω[8]

$$\Delta_t = \begin{cases} 0 \,, & b_{t-1} < b_t^* \,, \\ \delta_t \,, & b_{t-1} \geq b_t^* \,, \end{cases} \qquad b_t^* \sim \mathscr{B}^*(a_t, g_t, r_t) \,, \quad \delta_t \sim \Omega \,. \tag{5}$$

[6]In general, government consumption is not necessarily strictly non–productive. However since within this framework we do not study the optimal government policy we do not consider its impact on household decision strategy and welfare.

[7]Alternatively, we let ε_t^a to follow an empirical non-centred heavy-tailed distribution Γ established from the Slovak output gap data.

[8]Concerning the empirical distribution Ω of the stochastic default rate δ_t we refer to Bi and Leeper (2010) and Bi (2012). They computed the distribution from the sovereign debt defaults and restructures observed in the emerging market economies during the period of 1983 to 2005 since few sovereign default has been observed in developed countries in the post-war era. The cumulative distribution function of the default rate distribution is illustrated on Figure C.4 in the Appendix.

As noticed by Persson and Svensson (1989) and Alesina and Tabellini (1990) government expenditures are usually subject to political decisions that grow out of conflicts and compromises among parties with different ideologies. Therefore, to avoid explicit description of these political views following Bi and Leeper (2013) we specify the processes for government purchases and transfers to capture the trends and fluctuations of government expenditures observed in the data. Government purchase follows a simple autoregressive process

$$g_t = [g_{t-1}]^{\rho_g}\,[g]^{1-\rho_g}\exp\{\varepsilon_t^g\}\,, \qquad \varepsilon_t^g \sim \mathcal{N}(0,\sigma_g^2)\,. \tag{6}$$

Next, we assume that transfers evolve accordingly a Markov regime-switching process r_t driven by the constant transition matrix

$$P = \begin{pmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{pmatrix}.$$

Transfers have always explosive character: in each regime they grow exponentially with a time–dependent growth rate $\mu_t^{(i)}$. However, transfers in the risky regime ($r=2$) grow faster than in the no-policy-change regime ($r=1$). Therefore for a given regime of transfers it holds that

$$z_t = \begin{cases} \mu_t^{(1)} z_{t-1} + \varepsilon_t^z\,, & r_t = 1\,, \\ \mu_t^{(2)} z_{t-1} + \varepsilon_t^z\,, & r_t = 2\,. \end{cases} \qquad \varepsilon_t^z \sim \mathcal{N}(0,\sigma_z^2)\,. \tag{7}$$

We emphasize that requiring

$$\max_t \mu_t^{(i)}\beta < 1\,, \qquad i \in \{1,2\}\,, \tag{8}$$

guarantees for a discount factor β that transfers – though explosive in both regimes – remain square–summable in discounted expectations. This condition is inevitable to define correctly the fiscal limit.

Next, the government follows a simple Taylor–type rule that raises tax rate with adjustment parameter $\gamma > 0$ to retire the debt,

$$\tau_t = \tau + \gamma(b_t^d - b)\,. \tag{9}$$

Therefore, for any $\gamma > 0$ an equilibrium exist with non necessarily bounded debt – to ensure that debt is bounded in the steady state γ must be sufficiently large. Although government can raise tax rate freely, tax revenues do not exceed an upper bound imposed by a Laffer curve. Therefore, when transfers grow explosively, this rule does not prevent the government to default on its liabilities as explained in section 2.1.

The economy is populated by a continuum of infinitely-lived households living in the environment of complete insurance markets[9]. At each time t a representative household chooses

[9]In the complete asset markets, agents are able to purchase perfect insurance against the realisation of shocks. However, as Aiyagari (1994) and Aiyagari and McGrattan (1998) noticed in the incomplete insurance markets models households suffer such insurance protection against the realization of an idiosyncratic shock. Therefore, though all households are identical ex-ante, their inability to insure against the household-specific shock makes them heterogeneous ex- post.

consumption c_t, labour supply h_t, and bond purchases b_t that would maximize their utility[10]

$$\max \mathbb{E}_t \sum_{k=0}^{\infty} \beta^k U(c_{t+k}, h_{t+k}), \qquad U(c_t, h_t) = \log c_t + \phi \log(1 - h_t),$$

subject to the intertemporal budget constraint

$$(1 - \tau_t) a_t h_t - c_t + z_t = q_t b_t - b_t^d. \tag{10}$$

with transfers z_t, tax rate τ_t and debt default rate Δ_t taken as given. The parameter $\beta \in (0, 1)$ is the constant discount factor and ϕ is the representative household leisure preference parameter[11]. The household utility function is strictly concave and strictly increasing in leisure and consumption. Moreover, we assume that households consider the historical information captured in the empirical distribution Ω when pricing sovereign bonds. Therefore, the optimal allocation of resources requires that the marginal rate of substitution between consumption and labour supply coincides with the after-tax wage and households price bonds taking into account their expectation about the next–period the probability and magnitude of sovereign default:

$$\phi \frac{c_t}{1 - h_t} = -\frac{\partial U / \partial h_t}{\partial U / \partial c_t} = a_t (1 - \tau_t), \tag{11a}$$

$$q_t = \beta \mathbb{E}_t \left[(1 - \Delta_{t+1}) \frac{\partial U / \partial c_{t+1}}{\partial U / \partial c_t} \right] = \beta \mathbb{E}_t \left[(1 - \Delta_{t+1}) \frac{c_t}{c_{t+1}} \right]. \tag{11b}$$

Finally, the optimal solution to the household's maximization problem must also satisfy the subsequent transversality condition

$$\lim_{j \to \infty} \mathbb{E}_t \left\{ \beta^{j+1} \frac{\partial U / \partial c_{t+j+1}}{\partial U / \partial c_t} (1 - \Delta_{t+j+1}) b_{t+j} \right\} = 0, \qquad \forall t \geq 0. \tag{11c}$$

2.1 Concept of the Endogenous Fiscal Limit Distribution

The fiscal limit is the maximum level of debt that the government is able to pay back, defined as the sum of discounted expected maximum primary surpluses in all future periods. The dynamic and stochastic nature of the Laffer curve and shock processes imply that the fiscal limit is stochastic, with a probability distribution that depends on all the features of the economy, including private sector behaviours, the nature of policy behaviour, and the properties of the random disturbances in the economy. Moreover, while the conditional distribution of the fiscal limit is determined using also the information about the initial state of the economy, the unconditional distribution of the fiscal limit abstracts from it.

The *fiscal limit* is understood as the maximum level of debt that the government is able to service. Technically, it is defined as the sum of the discounted maximum fiscal primary surplus in all future periods. It is the point at which, for economic or political reasons, the government can no longer adjust taxes and spending (government consumption and transfers to households) to

[10] Notice that government consumption (e.g. in form of public goods) does not enter household utility since we do not study government optimal policy.

[11] The parameter ϕ measures the household willingness to supply their labour services: households are less disposed to work if $\phi > 0$ is large. The exact value of ϕ is not calibrated but determined from the model steady state.

stabilize debt. Therefore, its idea is based on two key elements: expectations of future primary surpluses (given the current information set) and the notion of the Laffer curve. An estimate of the tax rate at which the peak of the Laffer curve is also obtained.

Given the persistence of exogenous disturbances, the fiscal limit depends on the stochastic discount rate that reflect a non-zero probability of default, and on random disturbances hitting the economy in the future. Furthermore, the state-dependent conditional distribution reflects also the current state of the economy (productivity level, regime and level of transfers, level of government purchase). Under otherwise stated in the following text we discuss the principle of the conditional distribution of the fiscal limit.

The concept of the fiscal limit arises from the subsequent idea. Denote the primary surplus

$$\varsigma_t = \tau_t a_t h_t - z_t - g_t \,,$$

and iterate the budget constraint (10) employing the definition of the post-default debt (3):

$$\begin{aligned} b_{t-1} &= \frac{q_t b_t + \varsigma_t}{1-\Delta_t} = \frac{\varsigma_t}{1-\Delta_t} + \frac{q_t}{1-\Delta_t}\mathbb{E}_t \frac{b^d_{t+1}}{1-\Delta_{t+1}} \\ &= \frac{\varsigma_t}{1-\Delta_t} + \frac{q_t}{1-\Delta_t}\mathbb{E}_t \left[\frac{\varsigma_{t+1}}{1-\Delta_{t+1}} + \frac{q_{t+1}}{1-\Delta_{t+1}} b_{t+1}\right] = \dots \\ &= \mathbb{E}_t \sum_{k=0}^{T} \left[\prod_{j=0}^{k} \widetilde{q}_{t+j}\right] \frac{\varsigma_{t+k}}{1-\Delta_{t+k}} + \mathbb{E}_t \prod_{j=0}^{T} \widetilde{q}_{t+j} b_{t+T} \,, \end{aligned} \tag{12}$$

with $\widetilde{q}_t$ is defined as follows:

$$\widetilde{q}_{t+j} \equiv \begin{cases} 1\,, & j=0 \\ \dfrac{q_{t+j-1}}{1-\Delta_{t+j-1}}\,, & j>0\,, \end{cases} \,. \tag{13}$$

Studying the relationship (12)–(13) one can make two key observations. Firstly, due to trasversality condition (11c), restriction on maximal growth rate of transfers (8) and the Euler equation (11b) the second term in the equation above tends to zero as T raises. Therefore, to achieve the largest possible post-default debt b^d_t we need to maximize the present value of the sum of the current and all future primary surpluses. Secondly, to estimate the post-default debt b^d_t one must determine with the stochastic discount rate $\widetilde{q}_t$ that takes into account the possibility of future default.

Thus, essentially *the fiscal limit distribution cannot be determined independently from the default risk premium, these problems are coupled* – and this is the crucial difference from our original approach (see Mucka (2015)). Note that the default interest rate q_t arises endogenously as the solution to the subsequent nonlinear problem

$$\frac{(1-\Delta_t)b_{t-1} + g_t + z_t - \tau_t a_t h_t}{b_t} = q_t \tag{14}$$

$$q_t = \beta \mathbf{E}_t \left\{ [1-\Delta_{t+1}] \frac{c_t}{c_{t+1}} \right\} \tag{15}$$

$$\Delta_t = \begin{cases} 0\,, & b_{t-1} < b^*_t\,, \quad b^*_t \sim \mathscr{B}^*_t \\ \delta_t\,, & b_{t-1} \geq b^*_t\,, \quad \delta_t \sim \Omega\,, \end{cases} \tag{16}$$

where the tax rate prescription follows a standard Taylor–like rule (9). Furthermore, given the tax rate, the consumption-labour decision of households arises from household first order conditions (11a) and firms production technology (1).

2.2 The Laffer Curve

From the fiscal perspective, an increase in the proportional labour tax rate may or may not induce growth of tax revenues. This is the basis of the concept of the Laffer curve. Obviously, the point(s) on the curve where the tax revenues (measured as a function of tax rate) are at a maximum are particularly interesting. Within the baseline model where technology and government purchase follow standard autoregressive processes, there is a unique mapping between the state of economy characterised by the of technology and government purchases (a_t, g_t) and the tax rate $\tau_t^{\max}$ maximizing the collection of tax revenues $\Theta_t^{\max}$ given the state of the economy. Therefore, taking the current regime of transfers (political decision of the government), technology and government purchase as given, there is a unique maximum primary budget surplus and hence a *fiscal limit* defining the limit to the government's ability to service its debt. The fiscal limit distribution reports the probability that a particular debt level can be supported by taxing income at the peak of the Laffer curve, given the stochastic processes for transfers, government purchases, and productivity.

To estimate the country's fiscal limit, defined as the sum of the expected discounted maximum primary surplus, we need to maximize the difference between the maximum tax revenues[12] and expenditures (transfers and government purchases) in all future periods. We firstly consider the conditional, or state-dependent distribution of the fiscal limit,

$$\begin{aligned} \mathscr{B}_t^*(a_t, g_t, r_t, z_{t-1}) &\sim \mathbb{E}_t \sum_{k=0}^{\infty} \left[\prod_{j=1}^{k} \frac{q_{t+j-1}}{1-\Delta_{t+j-1}} \right] \frac{1}{1-\Delta_{t+k}} \varsigma^{\max}(a_{t+k}, g_{t+k}, r_{t+k}, z_{t+k}), \\ \varsigma^{\max}(a_{t+k}, g_{t+k}, r_{t+k}, z_{t+k}) &= \Theta^{\max}(a_{t+k}, g_{t+k}) - g_{t+k}(a_{t+k}, \varepsilon_{t+k}^{g}) - z(r_{t+k}, a_{t+k}, \varepsilon_{t+k}^{z}), \\ \Theta_t^{\max} &= (1+2\phi)\, a_t - \phi g_t - 2\sqrt{(1+\phi)\,\phi a_t\,(a_t - g_t)}\,. \end{aligned} \tag{17}$$

A conditional distribution emphasizes that bondholders expectations of repayment depend on the current state of the economy, including the policy regime and realisation of shocks. This approach is useful to assess the efficacy of fiscal reforms, and behaviour of the default risk premium in the short run[13]. However, for some analyses, particularly of long-run fiscal reforms, the unconditional fiscal limit distribution is more appropriate[14]. The current state of economy is less significant when determining the government's ability to service its debt in the long run. Therefore, the unconditional distribution $\mathscr{B}^*$ is not time varying and is given as:

$$\mathscr{B}_t^* \sim \mathbb{E}_t \sum_{k=0}^{\infty} \left[\prod_{j=1}^{k} \frac{q_{t+j-1}}{1-\Delta_{t+j-1}} \right] \frac{1}{1-\Delta_{t+k}} \left[\Theta^{\max} - g_{t+k} - z_{t+k}(r_{t+k})\right]. \tag{18}$$

[12]The existence and uniqueness of the revenues maximising tax rate $\tau_t^{\max}$ exists is guaranteed and is given as $\tau_t^{\max} = 1+\phi - \sqrt{(1+\phi)\,\phi\,(a_t - g_t)\,/a_t}$.

[13]Bi and Leeper (2013) used their model to study debt crisis and the effects of fiscal reforms in Greece.

[14]Bi and Leeper (2010) used the unconditional distribution of fiscal limits to understand the long-run impacts of fiscal reforms in Sweden.

2.3 Coupled Problems

We emphasize that due to stochastic default discount rate, fiscal limits distribution and the default rate cannot be determined separately, so the model must be solved at once.

However, we introduce an efficient scheme that enables us to find these two unknowns iteratively. Within each step we firstly calculate the fiscal limit distribution using the equations (2), (6), (7) and (17) taking the default bond price q and the previous step fiscal limit distribution as given. Then, using the updated fiscal limit distribution we solve numerically the nonlinear model prescribed by equations (1), (9), (11a) and (14)–(16) to obtain the more precise judgement of the default bond price q. We repeat this procedure until the convergence of both the fiscal limit distribution and the default bond price is achieved. Finally we derive the default risk premium. The procedure in discussed in depth in Appendix B.

2.3.1 Fiscal Limit Distribution

Inasmuch as there exists s unique mapping between the exogenous (a_t, g_t) and the revenue maximizing tax rate $\tau_t^{\max}$, to obtain the distribution of the fiscal limit we employ the Markov-Chain Monte-Carlo simulation technique. We firstly describe the procedure of determining the conditional distribution of the fiscal limits.

On each grid point of the discretized model state-space (a_t, g_t, r_t, z_{t-1}) we randomly draw $N = 10^6$ series shocks for the productivity, government purchase, transfers and regime of transfers of length[15] $T = 200$ given their distribution functions. Furthermore, to estimate correctly the stochastic default discount rate, we employ the default debt price $q_t = q(b_t^d, a_t, g_t, z_{t-1}, r_t)$ arising as the solution to the non-linear system (14)–(16) in the previous iteration. However, this approach implies the recursive formulation of the discount rate and – which is more important – to obtain the conditional distribution of the fiscal limit, one must solve the subsequent implicit highly non-linear equation

$$b_{t-1} = \mathbb{E}_t \sum_{k=0}^{T} \chi_{t+k} \frac{\varsigma_{t+k}^{\max}}{1-\Delta_{t+k}}, \qquad \chi_{t+k} = \begin{cases} 1, & k=0, \\ \chi_{t+k-1} \dfrac{q_{t+k-1}}{1-\Delta_{t+k-1}}, & k>0, \end{cases} \tag{19}$$

at each point (a_t, g_t, r_t, z_{t-1}) of the discretized state-space and for any draws of shock series $\{\varepsilon_{t+k}^a\}_{k=1}^T$, $\{\varepsilon_{t+k}^g\}_{k=1}^T$, $\{\varepsilon_{t+k}^z\}_{k=1}^T$ and $\{\varepsilon_{t+k}^r\}_{k=1}^T$. The reason is evident: since

$$q_{t+k} = q_{t+k}(b_{t+k}^d, a_{t+k}, g_{t+k}, z_{t+k-1}, r_{t+k}), \quad \text{and} \quad \Delta_{t+k} = \Delta_{t+k}(b_{t+k-1},, a_{t+k}, g_{t+k}, z_{t+k-1}, r_{t+k}), \tag{20}$$

both the actual default rate[16] Δ_t and the bond price q_t depend on the begin-of-period debt b_{t-1} i.e. the present value of the sum of current & all future maximal surpluses (the right hand side of the equation (19)) is for the given grid point (a_t, g_t, r_t, z_{t-1}) and the associated draws of shock series the function of the time t begin-of-period debt b_{t-1} only (the left hand side of the equation (19)).

Technical details of the procedure can be found in Appendix B.2.

[15]The simulation period approximating the infinite time horizon for this quarterly model is determined by the length of the projections for the growth rates of transfers, 50 years.

[16]Here we relax from the the originally stochastic maximal default rate $\delta_t \sim \Omega$ and prefer its mean value δ.

Fiscal Limits in the Long-Run. The unconditional distribution of the fiscal limits is useful particularly when we study the long-run impact of the reforms on the sustainability of public finance. Since the unconditional distribution is independent of the current state of the economy, the actual default rate is a function of is a function of the begin-of-period debt only. This fact simplifies[17] significantly the calculation of the endogenous discount rate and thus speeds-up the estimation of the unconditional fiscal limit. For further details about the estimation of unconditional fiscal limit distribution we refer the reader to Appendix B.4.

2.3.2 Debt Pricing Rule and the Default Risk Premium

Following the iterative procedure within the second phase of each iteration we need to solve the system of equations (14)–(16) given the approximation of the fiscal limit distribution obtained in the first phase of the iteration.

Thus, the debt pricing rule associated with the pre-default debt is determined by solving the nonlinear forward–looking model (14)–(15) assuming that tax rate follows a simple Taylor–like rule (9) and the actual default rate satisfies (16). Given the estimated fiscal limit distribution we solve this problem numerically employing the monotone mapping method (see Coleman (1991) and Davig (2004)). In each step of the iteration we map the current state ψ_t to obtain the updated guess of the debt rule (i.e. the future begin-of-period debt level) $b_t(\psi_t)$ by solving the core equation of the model,

$$\frac{(1-\Delta_t)b_{t-1}+g_t(\psi_t)+z_t(\psi_t)-\tau_t a_t h_t(\psi_t)}{b_t(\psi_t)} = \beta \mathbf{E}_t \left\{ [1-\Delta_{t+1}(\psi_{t+1})] \frac{c_t(\psi_t)}{c_{t+1}(\psi_{t+1})} \right\}, \tag{21}$$

employing the algorithm of Sims (1999)[18]. The right-hand side of the formula (21) is evaluated by applying various numerical quadrature methods[19] and interpolation techniques[20] (see Appendix B.1).

After obtaining the decision rules for each point in the discrete state–space, we employ the budget constraint to find the iteration-specific approximation of the bond-pricing rule, q_t. For more technical details about the whole procedure we refer the reader to Appendix B.3.

Providing that both the fiscal limit distribution and pricing rule converge (i.e. the differences between the distributions and rules obtained in the current and past iterations are sufficiently

[17]In case of the unconditional distribution $\Delta_{t+k} = \Delta(b_{t+k-1})$ Therefore the projection onto the actual default rate grid Δ is reduced to a one-dimensional projection. This cuts the problem complexity and in each iteration makes the determination of the distribution and the associated bond price easier.

[18]Unfortunately, there is no guarantee that this algorithm is able to find the solution to (B.6) for every possible parametrization such that it lives within some reasonable boundaries. Therefore, the model is extended to include some built–in approximation techniques.

[19]We combine various numerical quadrature methods to approximate the right-hand of the problem (21) i.e. the multiple integral: the Simpson's 3/8 rule (Simpson's second rule) with equidistant points in two dimensions and trapezoidal rule (in one dimension) (Jeffery J. Leader (2004)) and in order to lower the error we also employ the Gauss-Legendre quadrature with non-equidistant nodes (Golub and Welsch (1969)). Further details are deeply discussed in Appendix B.1 and B.3.

[20]We use cubic splines to perform a smooth interpolation between grid points. This leads to a significant reduction of the error - in compare to standard linear interpolation.

small) the iterative algorithm terminates and we evaluate the default risk premium,

$$r_t = R_t - R_t^f = 1/q_t - 1/q_t^{\Delta_t=0}. \tag{22}$$

Otherwise, the last approximation of the bond pricing rule is passed to the first phase of the subsequent iteration of the procedure.

2.4 Discussion

As noticed by Bi and Leeper (2013) in linearised models with stationary transfers a Taylor-like tax rule (9) that increase tax rate whenever the debt rise, can prevent debt explosion unless the tax adjustment parameter is too small. Debt stabilisation is not guaranteed when the tax rate approaches the peak of the Laffer curve or in case of explosive transfers. Even if the average tax rate is far from the peak of the Laffer curve, rising transfers cause debt growth and force the government following the Taylor-like rule (9) to increase then tax rate. In this environment debt may gradually augment to such a level that government will be unable to repay it in full although it levies taxes following a standard debt-stabilising Taylor-like rule (9). Forward-looking risk-averse agents consider a positive probability of eventually hitting the peak of the Laffer curve and a potential default in the future when pricing current government liabilities. Hence, even if the current tax rate is well below the peak of the Laffer curve, this can incite concerns for debt sustainability and sovereign default.

3 Benchmark Calibration

In order to calibrate the model on quarterly frequency we use the 2013–2060 projections of Slovak data as published in Council for Budget Responsibility (2014).

Government purchase covers government final consumption of expenditures, subsidies, public wage bill and net capital transfers and in the steady-state covers 16.4 percent of the GDP. The average tax rate is defined as the ratio of the total tax revenue over the GDP, including social security and indirect and direct taxes and is consistent with 40 percent steady state debt–to–GDP and the annual discount rate 0.98. As a consequence, the steady state rate attains 33.04 percent (see Appendix C.2) which is higher than the tax rate 31.68 percent observed in the data.

The leisure preference parameter ϕ set such that households spend 25 percent of their time by working (leisure is 75 percent of their time) and the Frisch elasticity of labour supply is 3. Furthermore, we assume that productivity is unity in the steady state.

The coefficients affecting the model dynamics are obtained employing the Bayesian approach. Using the 2000–2015 time series for the transfers, government purchase, public debt and effective labour tax rate, we estimate relatively small sensitivity of tax rate to past debt deviation, $\gamma = 0.0483$. Next, concerning the productivity persistence and volatility we find out that data confirm the recent transitory character of the Slovak small open economy that has faced many structural breaks: persistence of technology is relatively small in compare to developed economies, $\rho_a = 0.7664$ while standard deviation of shocks remains considerably high, $\sigma_a = 0.0167$. On the other hand side, government consumption evolution remains relative stable with $\rho_g = 0.8329$ and standard deviations of fiscal shocks are moderate, since $\sigma_g = 0.0120$ and $\sigma_z = 0.0092$ as summarised in Table 3.1. For further details about historical data and Bayesian priors and posteriors see Appendix C.2.

Table 3.1 : Benchmark calibration

Equilibrium debt/GDP (in %)	b/y	40
Initial level of transfers/GDP (in %)	z/y	18.6
Initial level of government consumption/GDP (in %)	g/y	16.4
Annual discount rate	β	0.98
Frisch elasticity of labour supply	$1/h-1$	3
Transfers: average annual growth rate (benchmark scenario)	$\overline{\mu}_1$	0.25%
Transfers: average annual growth rate (risky scenario)	$\overline{\mu}_2$	0.31%
Probability of remaining transfers in benchmark scenario	p_1	0.96875
Probability of remaining transfers in risky scenario	p_2	0.96875
Technology persistence	ρ_a	0.7664
Government consumption persistence	ρ_g	0.8329
Tax sensitivity to past debt deviation	γ	0.0483
Technology shocks standard deviation	σ_a	0.0167
Government consumption shocks standard deviation	σ_g	0.0120
Transfers shocks standard deviation	σ_z	0.0092

Furthermore, referring to the average length of the political cycle in Slovakia we assume that transfers reside in each regime for 8 years.

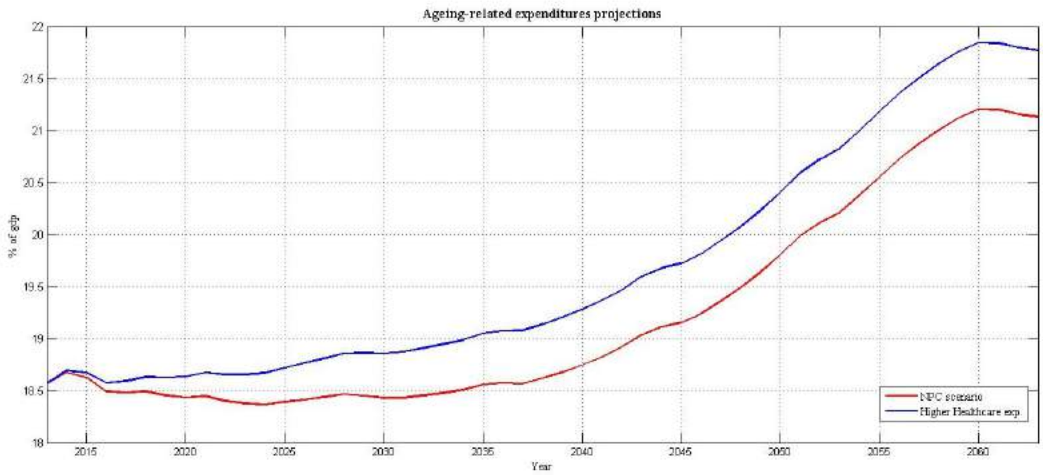
Concerning the model calibration there are two issues that need to be explained much deeper: the character of transfers and specifics of the business cycle in Slovakia.

3.1 Explosive Transfers

The benchmark scenario of the long-term development of public finances, as defined by the Fiscal Responsibility Act, is developed by merging the medium-term scenario with long-term projections of the revenues and expenditures sensitive to population ageing and by incorporating other implicit and contingent liabilities (old-age, armed forces and disability pensions; healthcare, long–term care; education and unemployment benefits). Therefore, transfers include social security payments and material social transfers (transfers in kind), which means all demography structure sensitive government payments. Besides the projections of the fundamental demographic shifts expected in the next 50 years inducing increasing share of ageing–related government expenses on GDP even in the *no policy change* scenario (see Council for Budget Responsibility (2014)), the government may adopt additional long–term measures that adjust (increase or cut) the ageing–related expenses.

Under the baseline scenario between 2013–2060 a more than 13.8 percent increase in expenditures sensitive to population ageing (from 18.6 percent to more than 21 percent of GDP, see Figure 3.1) occurs which leads to average annual growth rate 1.0025. Alternatively, in the scenario with higher healthcare expenditures the share of transfers to GDP increases to more than 21.75 percent of GDP (average annual growth rate 1.0031).

Figure 3.1 : Projections of ageing related government expenditures



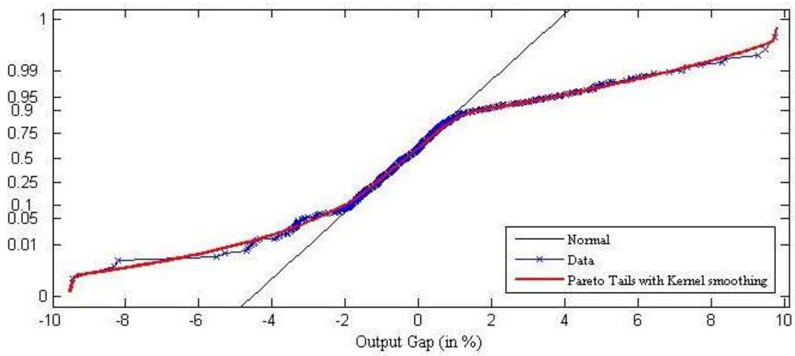
Ageing-related expenditures projections: red line corresponds to the no-policy-change scenario, blue line corresponds to the risky scenario with higher healthcare expenses.

3.2 Business Cycle Distribution

The technology is perceived in much broader sense in order to describe the possible fluctuations in real exchange rate and to emphasize the implicit impact of the foreign economy on domestic environment that may be particularly relevant in case of the small open economy. The calibration of technology process parameters need to reflect the volatility and autocorrelation of the domestic output.

Therefore, to determine the distribution of the business cycle for Slovakia – a small open economy with short history, many structural breaks and changes in methodologies concerning the relevant data – we employ available estimates of the output gap. To overcome the uncertainty arising from the short time series and volatile data and increase the robustness of the output gap distribution estimation we consider all output gap time series for Slovakia published by several domestic and international[21] institutions as well as results obtained using standard filtering techniques[22] between 2000–2014[23]. Using this pragmatic approach we solve the lack–of–data issue and minimise the problems associated with small open economies and filtering techniques[24]. However, relatively low technology persistence ($\rho_a = 0.7664$) and high volatility ($\sigma_a = 0.0167$) are consistent with observations of Neumeyer and Perri (2005) and Aguiar and Gopinath (2007) on the character of the business cycles in less developed economies.

Figure 3.2 : Output gap data distribution



Comparison of the output gap data distribution (blue line with markers) to the normal distribution (black dashed line) and empirical Pareto-tailed kernel smoothing distribution (red thick line).

Inspecting the Slovak output gap data we find that extreme cases are not rare and the probability of keeping the output gap close to its mean decays rapidly[25]. Therefore, to model frequent structural breaks, we first use *Pareto tails* and the *kernel smoothing procedure* to estimate

[21] Slovak Ministry of Finance, National Bank of Slovakia, European Commission, Bank for International Settlement.

[22] Hodrick–Prescott filter, multivariate Kalman filter, and Principal component analysis.

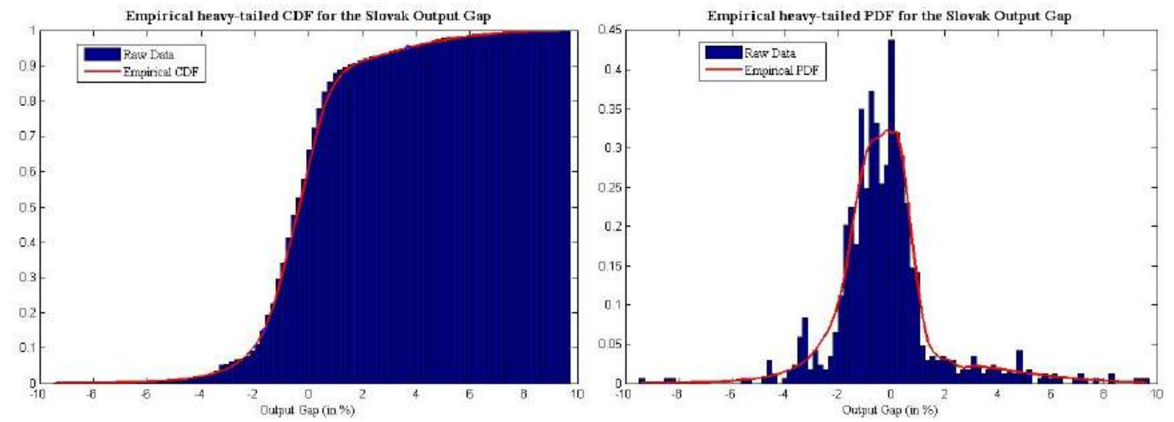
[23] Since these time series are usually on annual frequencies we interpolate them to obtain the approximations of quarterly output gap data.

[24] As shown in Odor and Jurasekova-Kucserova (2014) the current benchmark method in Europe based on the production function approach has in our view a lot of shortcomings in small and open economies – short-time series to estimate long-term trends with many structural breaks, high uncertainty around capital stock estimates, downplaying international capital and labour mobility, size of current account imbalances and banking sectors relative to GDP can be important in small and open economies, frequent supply side shocks, end-point problem of the HP-filter.

[25] Table C.3 in the Appendix C.3 contains the detailed descriptive statistic of the collected output gap data.

the distribution between these fat tails[26] (properties of this empirical distribution summarises in Appendix C.3). In the simulation of the fiscal limit, we draw random technology shock series employing this empirical distribution.

Figure 3.3 : Empirical distribution of the Slovak business cycle



Empirical distribution of the Slovak business cycle as estimated from 2000-2014 data taken from Slovak Ministry of Finance, National Bank of Slovakia, European Commission, Bank for International Settlement and Hodrick–Prescott filter, multivariate Kalman filter, and Principal component analysis. Asymmetric fat tails (quantiles 0.15 and 0.95) are approximated by Pareto distribution whereas the Kernel smoothing procedure is employ to estimate the distribution between the tails.

[26] Alternatively, we employ the *location-scaled t-distribution* to model the whole asymmetric heavy–tailed distribution of the Slovak business cycle.

4 Results

To understand the impact of various model parameters we proceed as follows. We start with a baseline case with only productivity and government consumption shocks and let transfers to follow the no-policy-change path. We then modify one parameter at a time, while keeping all other parameters the same as in the baseline case to understand the quantitative impact of macroeconomic fundamentals upon the distribution of fiscal limits and the behaviour of the associated default risk premium.

The state-dependent fiscal limit distributions result from the approach introduced in Section 2.3 and described in detail in Appendix B. This iterative method is employed in order to solve the coupled problem and determine simultaneously both the fiscal limit distribution and the stochastic default discount rate (from which we can derive directly the default risk premium). To obtain the time-dependent distribution we employ the Markov-Chain Monte-Carlo simulation technique and solve the recursive implicit problem (19) on discrete state space (a_t, g_t, r_t, z_{t-1}).

From the technical point o view, using non-equally spaced grid nodes and preferring cubic splines when interpolating between grid points leads to a significant reduction of the numerical error. Similarly, we combine various numerical quadrature methods when evaluating the stochastic default discount rate. Appendix C.5 contains the specification of grids and the numerical quadrature techniques employed when solving the problem (19).

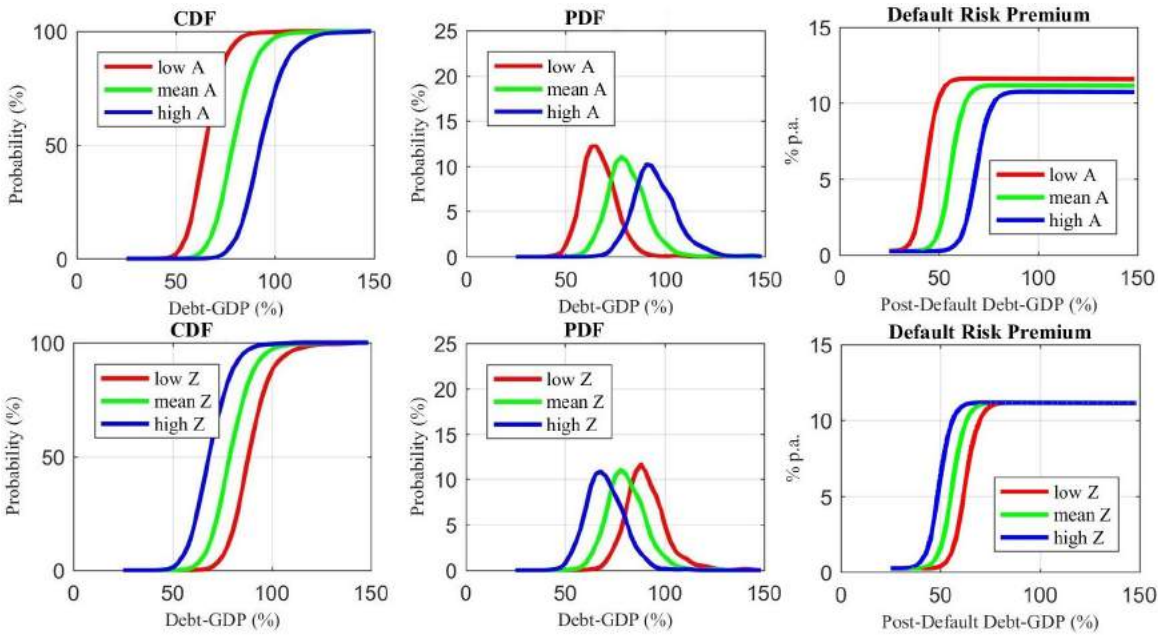
4.1 Endogenous Fiscal Limit Distribution

In the baseline case we assume that the economy faces productivity and transfers shocks and transfers switch between the no-policy-change and risky scenarios. Note that the transition matrix reflects country's usual eight-year political cycle. Furthermore, for the sake of simplicity here we do not consider the impact of the business cycle on transfers although there is a strong empirical evidence on countercyclical character of transfers.

The left panels in the Figure 4.1 (or Figure D.1 in the Appendix D) show the probability distribution functions (PDFs) and the corresponding cumulative distribution functions (CDFs). Thus, the CDFs can be understood as the *probability of sovereign default at different debt levels*. Next, the right panels depict the country's debt price and the associated default risk premium derived for various post-default debt levels. Furthermore, the top panels compare the state-dependent distributions, prices and premiums at different productivity levels representing a sudden fall or increase in country's output by 10 percent of its steady-state level (which is $\pm 6\sigma_a$) while having transfers at their equilibrium levels and following the no-policy-change growth path (NPC). Similarly, the bottom panels in the Figure 4.1 present this comparison providing that transfers rise or decline by 3.68 percent (which is $\pm 4\sigma_z$) assuming the equilibrium level of productivity and NPC scenario. Notice that with such high initial level of transfers, under the no policy change scenario in 2060 the share of expenditures sensitive to population ageing to GDP exceeds 25 percent of GDP – and this is essentially more than the expected share of age-related expenses to GDP assuming the current level of transfers following the risky growth path.

It is evident that the impact of the levels of productivity and transfers on the distribution of the fiscal limit and the default risk premium is significant and moderate debt levels may be associated with a considerable chance of default. Concretely, although in case that the debt reaches 60 percent of the output the chance of default does not exceed 9 percent in normal times and medium level of transfers, an extreme 10 percent productivity fall leads to even 39 percent chance of default and 3.68 percent increase in the initial level of transfers implies 28 percent chance of default. Concerning the default risk premium, in normal times investors penalise the country which debt attains the Maastricht criterion by approximately 8 p.p. premium. However, during bad times they ask for 12 p.p. premium. A similar premium is required when transfers are too high.

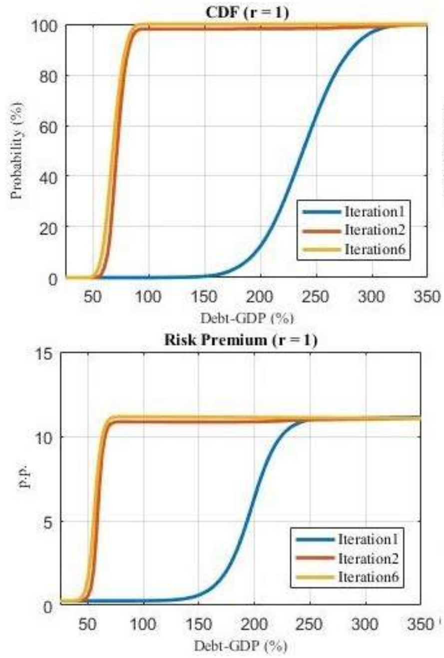
Figure 4.1 : Fiscal limit distribution (baseline scenario)



Endogenous distribution functions and default risk premium estimated iteratively for various levels of productivity (upper row) and transfers (lower row) in the baseline case under the no-policy-change scenario.

Convergence. The convergence of iterative method employed in order to solve the coupled problem and determine simultaneously both the fiscal limit distribution and the stochastic default discount rate is obtained after 6 iterations with the convergence error 10^{-6} as illustrated on Figure 4.2. The major shift in the distribution (and also the risk premium) comes with the second iteration (see the left panel Figure 4.2). This is obvious – although the very first approximation of the distribution is obtained when the endogenous stochastic default price coincides with the risk-free discount rate β, the first approximation of the debt price (and so the risk premium) is essentially lower for high debt (see Figure D.10 in Appendix D). However, the need for adjustments in the fiscal limit distribution and debt price falls sharply with each subsequent iteration and the stabilization (and hence the convergence) comes after 6 iterations.

Figure 4.2 : Convergence of the iterative method



Approximations of the cumulative distribution function of the fiscal limit and the default risk premium in the baseline scenario for the equilibrium path (starting from normal times with average level of transfers and NPC growth rate). Figures on left panel depict the shift in the solution approximation between the first and second iteration. Figures on right panel show the procedure stabilisation and gradual convergence during the subsequent iterations.

4.2 Quantitative Analysis

In the forthcoming text we scrutinize the impact of changes in selected model parameters on the fiscal limit distribution. Although the very first sensitivity analysis was presented in our recent paper (see Mucka (2015)) this work is focused on study of various realistic macro and fiscal scenarios. Concretely, we investigate how changes in sensitivity of tax rate to debt deviation, sizes of shocks, growth rate of transfers, fiscal policy credibility, and risk-free discount rate affect the endogenous fiscal limit distribution and the associated default risk premium. Description of selected case studies is summarised in Table 4.1.

Table 4.1 : Sensitivity analysis: specification of case studies

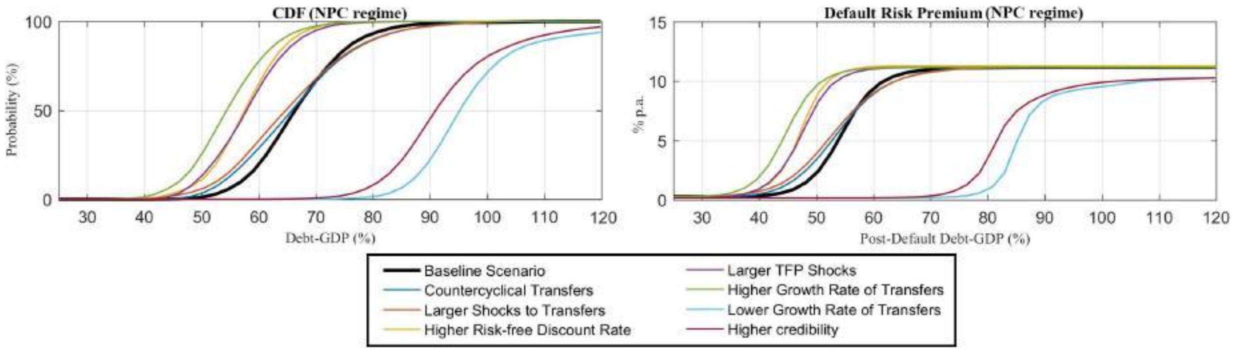
Scenarios	β	$\overline{\mu}_1^4$	$\overline{\mu}_2^4$	σ_a	σ_z	γ	p_2
Baseline	0.98	1.0026	1.0032	0.0273	0.0168	0.0201	0.0313
Higher growth rate of transfers	0.98	**1.0029**	**1.0035**	0.0273	0.0168	0.0201	0.0313
More credible fiscal policy	0.98	1.0026	1.0032	0.0273	0.0168	0.0201	**0**
More volatile transfers	0.98	1.0026	1.0032	0.0273	**0.0202**	0.0201	0.0313
Higher risk-free interest rate	**0.97**	1.0026	1.0032	0.0273	0.0168	0.0201	0.0313
More volatile productivity	0.98	1.0026	1.0032	**0.0328**	0.0168	0.0201	0.0313
Higher responsiveness of tax rate	0.98	1.0026	1.0032	0.0273	0.0168	**0.0242**	0.0313

Fiscal Policy Credibility We study a case of credible reform in which transfers are more likely to follow the no-policy-change path than in the baseline scenario. Concretely, the higher is the regime-switching probability p_1 the less likely transfers will switch from the less explosive to more explosive regime and thus improve the debt sustainability. To illustrate this situation we consider a credible reform that raises the expected duration of the NPC regime on 16 percent, the chance of default is approximately means that once transfers follows the NPC path, the probability of renege on the fiscal reform and reverting to the risky regime next period is only 1.56 percent.
The impact of that commitment on fiscal sustainability is essential: the extremely low chance to default associated with the Maastricht criterion and only 18 percent probability to default when debt attains 80 percent of output. Such a significant shift in the fiscal limits distribution is caused by reduced expected future primary deficits (caused by lower transfers) additionally boosted up by lower risk premium asked by investors. Therefore, very credible fiscal reforms can improve fiscal sustainability and reduce debt service fundamentally.

Transfers Growth Rate. As another alternative, we examine for the situation when age-related transfers in both regimes grow by 10 percent faster than in the baseline case. It means that the average growth rate of transfers in the no-policy-change regime attains 0.29 percent on annual basis and on 50-year horizon leads to an additional increase on spending by more than 1.3 percent of GDP. Likewise, in risky regime the average growth rate achieves 0.35 percent and after 50 years it implies extra expenditures on transfers of size 1.6 percent of GDP. More aggressive growth of transfers leads to higher chance of country default even with a relatively small debt as it projects considerably larger expected future expenditures when debt attains 60 percent of GDP the probability of default in normal times raises to 30 percent and even doubles during bad times (see Figure D.2 in Appendix D).
On the other side, consider the situation in which the government commits to reduce the growth rate of age-related transfers in both regimes by 10 percent. It means that if transfers follow the no-policy-change regime their grows rate attains 0.23 percent on annual basis and a significant reduction in expected expenditures as after 50 years the share of are-related transfers on GDP is 20.9 percent (by 0.3 percent of GDP less than in the benchmark case). The impact of such a fiscal reform on public finance sustainability is evident, since in normal times there is no chance of default associated with the Maastricht debt level. Moreover, while in the benchmark scenario 90 percent debt/GDP ratio implies the default, the reduction in growth rate of transfers causes that the chance of default is only approximately 25 percent. Such a significant shift in the fiscal limits distribution is caused by reduced expected future primary deficits (caused by lower transfers) additionally boosted up by lower risk premium asked by investors. Therefore, fiscal reforms that decrease the growth rate of sge-related expenses can improve fiscal sustainability and reduce debt service fundamentally.

Tax Rate Sensitivity. Providing that the government tends to use taxes in order to stabilize the excessive debt more than in the baseline scenario, i.e. raises tax rate by 0.024 p.p. whenever the post-default debt deviate from its equilibrium level by 1 p.p. it directly affects behaviour of households and firms. Increasing debt mitigates labour supply, reduces production, lowers bond price and hence implies more expensive debt service. Thus, higher chance of default even

Figure 4.3 : Sensitivity analysis: fiscal limit distribution and risk premium



Comparison of the cumulative distribution functions of the fiscal limit and the corresponding default risk premium estimated for various scenarios and equilibrium initial conditions.

for a relatively low debt levels (with 60 percent debt/GDP the probability of default in normal times raises to 15 percent – see Figure D.7 in Appendix D) is obvious.

Risk-free Interest Rate. Since the economy has no monetary policy, lower time-discount rate β can be viewed as the exogenous raise of baseline risk-free interest rate. From Figure 4.3 and Figure D.6 (Appendix D) it can be seen that higher risk-free rate by 1 p.p. may lead to a significant increase of the country's chance of default even for a relatively low debt levels when debt attains 60 percent of GDP the probability of default in normal times raises to approximately 15 percent. The influence of changes on the distribution of the fiscal limit and the corresponding default risk premium is evident since higher interest rate inhibits current activity of households and firms and makes debt financing more expensive.

Volatility of Productivity and Transfers Shocks. Finally, with higher volatility of the business cycle or transfers, the distributions become more heavy tailed and shift to the left with increasing shock volatility. This is obvious, since the effects on any shock in a highly persistent process last longer.

4.3 Discount Fast Future Income

Referring to (17) the stochastic discount rate is employed to determine the fiscal limit as the sum of the expected discounted maximum primary surpluses in all future periods. The problem of finding the fiscal limit distribution, default rate and the linkage between them - stochastic discount rate - must be solved at once. However, this is very work-intensive and computationally expensive and must be recalculated with any minor change of model parameters.

Therefore, our aim is to approximate the stochastic discount rate by a constant β^* and use this *true beta* in various policy simulation. The major advantage of this approach is that to simulate different policy case-studies with constant endogenously determined discount rate β^* instead of the highly non-linear implicit (17) we may use its subsequent simplified explicit version[27]

$$\mathscr{B}_t^* \sim \mathbf{E}_t \sum_{k=0}^{\infty} (\beta^*)^k \frac{c_t}{c_{t+k}} \varsigma_{t+k}^{\max}, \tag{23}$$

where $\varsigma_{t+k}^{\max}$ is the maximal primary surplus at time $t+k$. Hence, using the endogenous fiscal limit distribution and bond price resulting from the iterative procedure for the baseline scenario we find that the endogenous stochastic discount rate is approximately equivalent to *true beta* attaining the value 0.9076 (see Table 4.2).

Table 4.2 : Finding *True beta*: iterative procedure

Iteration	1	2	3	4	5	6
True Beta approximation	0.98	0.9033	0.9111	0.9053	0.9092	0.9076

Convergence in the true beta estimation is obtained after six iterations of the algorithm.

4.4 Long-Run Implications

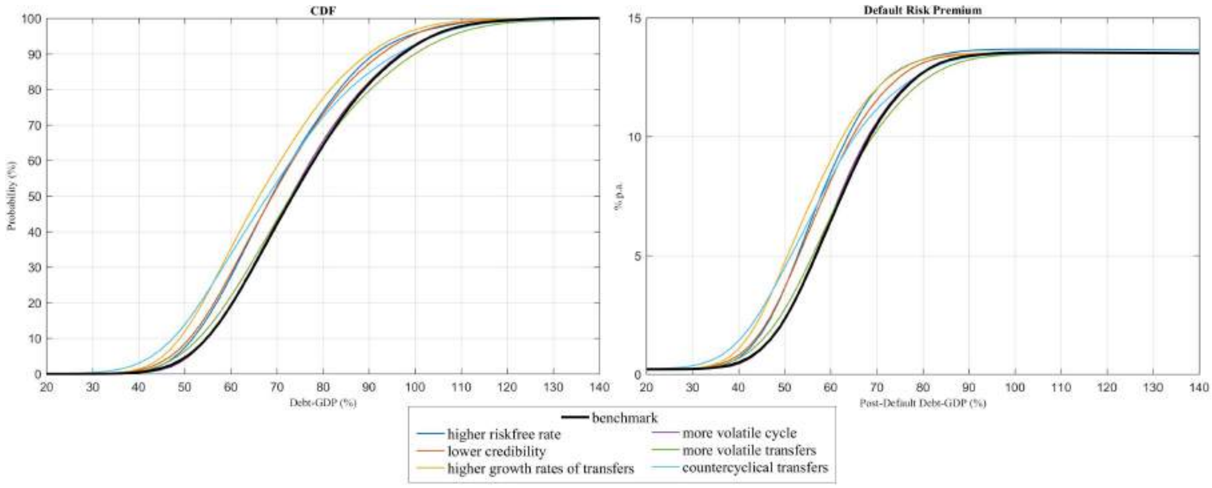
Recalling Section 2.2 the unconditional distribution of fiscal limit is particularly useful when long-run impacts of current reforms are analysed. The unconditional, state-independent distribution of the fiscal limits can be employed to study large and permanent changes in fiscal behaviour. Furthermore this measure represents an alternative approach to endogenous *true beta* determined in Section 4.3 to perform various policy simulations and case-studies.

Figure 4.4 illustrates the long-run implication of various changes in the model parameters – growth rate of age-related transfers, policy credibility, volatility of transfers and business cycle and risk-free discount rate value – on the country's chance of default and the corresponding default risk premium and compare them to the benchmark unconditional distribution[28] and default risk premium. Concerning the quantitative analysis of the unconditional distribution we follow the strategy proposed for the conditional distribution, so we vary the model parameters and subsequently study different scenarios accordingly to Table 4.1.

We observe that the benchmark long-run distribution is consistent with the fiscal limit distribution obtained for the economy in the equilibrium, although it is more disperse than its conditional counterpart. This is obvious, since the uncertainty about the initial state of the economy (the level of age-related exponentially growing transfers especially has strong consequences on agents' expectations about the future). Assuming the benchmark calibration of the model in which transfers do not reflect the current phase of the business cycle we find that that the long-run probability of default attains 20 percent when the sovereign debt attains 60 percent of the domestic output which is slightly higher than its conditional counterpart.
Next, from Figure 4.4 (or Figure D.11 in Appendix) it follows that besides changes in the exogenous monetary policy (varying risk-free rate) the essential impact of the expected fiscal policy

[27] This formulation is also used to determine the first approximation of the fiscal limit distribution.
[28] To obtain the unconditional distribution of the fiscal limit and the associated default risk premium we simulated the model 10^7 times for 200 years and dropped the first 100 simulations. The convergence of the procedure is guaranteed as $\beta \sup\{\mu\} < 1$. For further details on the distribution estimation we kindly refer the reader to see Appendix B.4.

Figure 4.4 : Unconditional distribution of fiscal limits



Comparison of the cumulative distribution functions of the fiscal limit unconditional distribution and the corresponding default risk premium estimated for various scenarios and equilibrium initial conditions.

on the unconditional distribution of fiscal limit are confirmed even in the long-run. Indeed, providing that transfers are countercyclical[29] the distribution of the fiscal limits – now very sensitive to realisations of productivity shocks – becomes very disperse and therefore the chance of default associated with the Maastricht debt limit raises to almost 35 percent. A likewise result is obtained if we assume that transfers in both regimes grow by 10 percent faster than in the baseline case[30]. Lower credible policy (i.e. when the government is more likely to let transfers to evolve following the risky regime) worsen the fiscal sustainability in the long-run and thus leads to approximately 30 percent chance of default associated with 60 percent debt/GDP.
Therefore our conclusion on the too benevolent Maastricht debt limit is valid in the long-run.

[29] Government tends to increase age-related transfers by 4 percent of their steady-state value with -1 percent output gap.

[30] It means that the average growth rate of transfers in the NPC regime attains 0.29 percent on annual basis and 0.35 percent on annual basis in case of the risky regime.

5 Concluding Remarks

We built a simple general equilibrium framework that is able to capture important aspects of the Slovak economic and fiscal policy environment such as heavy-tailed distribution of cyclical conditions, significant growth in demography-related spending, and counter-cyclicality of transfers.

We used this simple model to provide an estimate of the fiscal limit - the maximum serviceable level of public debt and a model-based estimate of the associated risk premium demanded by households aware of the fact that the government might default on its obligations. Owing to the feedback loop between the fiscal limit and the risk premium embedded in the model we derived the estimates of the fiscal limit and the risk premium simultaneously. Introducing such an interaction altered the shape of both the fiscal limit distribution as well as the levels of fiscal premia in a way that make these estimates look more realistic. We have shown that both are very sensitive to the rate of growth of transfers and their countercyclicity, business cycle turbulences and to credibility of fiscal policy pursued. Since these impacts are even amplified in the long-run, a proper and credible fiscal policy is crucial. Controlling age-related spending is thus a key task if the government wants to avoid financing difficulties. We have also demonstrated that the nature of economic conditions makes the distribution of the fiscal limit heavy-tailed, and – as a consequence – apparently safe levels (and legislated limits) of public debt might not be safe in reality.

The model we use can be developed further. A richer model could also provide a better account of the consequences of the cyclical nature of spending items. The first possible model extension reflect a significant openness of Slovak economy. However, obtaining the endogenous distributional of the fiscal limit with the model augmented by the economy openness feature even in a very educed form is essentially more computationally expensive.
The second model extension arises from the well-known fact that the RBC theory sees business cycle fluctuations as the efficient response to exogenous changes in the real economic environment. Therefore, the government should not intervene through discretionary fiscal or monetary policy designed to actively smooth out economic short-term fluctuations. Thus employing the New-Keynesian approach in sense of monopolistic competition and wage markup by adding the inefficient fluctuation element to the model can eliminate this model drawback.
Thirdly, in reality country's cost of default is not only reputational and consequently the economy faces a substantial loss in output. Therefore the model can incorporate the nonlinear costs of default. Next, in the current model setting the design of the tax rule implies increased real activity in the post-default period although defaults typically lead to a decline in output. So the tax rule should be adjusted to avoid this counterfactual result.
Finally, introducing idiosyncratic productivity shocks at household level may be plausible if welfare implications of debt limits are studied.
We leave all these extensions on our future research agenda. However, the key policy messages arising from this paper should continue to hold if not with a greater force.

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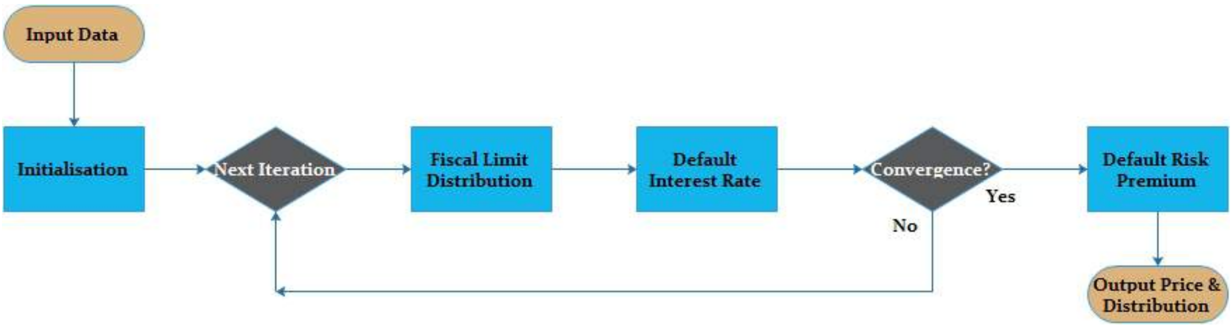
Appendix A Model Definition

$$
\begin{aligned}
a_t &= a_{t-1}^{\rho_a} a^{1-\rho_a} \exp\{\varepsilon_t^a\}, \qquad \varepsilon_t^a \sim \Gamma && \text{(A.1)}\\
g_t &= g_{t-1}^{\rho_g} g^{1-\rho_g} \exp\{\varepsilon_t^g\}, \qquad \varepsilon_t^g \sim \mathcal{N}(0,\sigma_g^2), && \text{(A.2)}\\
c_t &= \frac{(a_t - g_t)(1-\tau_t)}{1+\phi-\tau_t}, && \text{(A.3)}\\
h_t &= \frac{a_t(1-\tau_t)+\phi g_t}{a_t(1+\phi-\tau_t)}, && \text{(A.4)}\\
z_t &= \begin{cases} \mu_t^{(1)} z_{t-1} + \varepsilon_t^z, & r_t = 1, \\ \mu_t^{(2)} z_{t-1} + \varepsilon_t^z, & r_t = 2, \end{cases} \quad , \qquad \varepsilon_t^z \sim \mathcal{N}(0,\sigma_z^2), && \text{(A.5)}\\
\Theta_t^{\max} &= (1+2\phi)\, a_t - \phi g_t - 2\sqrt{(1+\phi)\,\phi a_t\,(a_t - g_t)}, && \text{(A.6)}\\
\varsigma_t^{\max} &= \Theta_{t+k}^{\max} - g_{t+k} - z_{t+k}(r_{t+k}), && \text{(A.7)}\\
\mathcal{B}_t^* &= \mathbb{E}_t \sum_{k=0}^{\infty} \left[\prod_{j=0}^{k-1} \frac{q_{t+j}}{1-\Delta_{t+j}}\right] \frac{\varsigma_{t+k}^{\max}}{1-\Delta_{t+k}}, && \text{(A.8)}\\
\tau_t &= \tau + \gamma[(1-\Delta_t)b_{t-1} - b], && \text{(A.9)}\\
q_t b_t &= (1-\Delta_t)b_{t-1} + g_t + z_t - \tau_t a_t h_t, && \text{(A.10)}\\
b_t^d &= b_{t-1}(1-\Delta_t) && \text{(A.11)}\\
q_t &= \beta \mathbf{E}_t \left\{ [1-\Delta_{t+1}] \frac{c_t}{c_{t+1}} \right\} && \text{(A.12)}\\
\Delta_t &= \begin{cases} 0, & b_{t-1} < b_t^*, \quad b_t^* \sim \mathcal{B}_t^* \\ \delta_t, & b_{t-1} \geq b_t^*, \quad \delta_t \sim \Omega, \end{cases} && \text{(A.13)}
\end{aligned}
$$

Appendix B The Concept of Coupled Problems

Referring to Section 2.3, due to stochastic character of the default discount rate, the fiscal limit distribution cannot be determined independently from the default risk premium. These problems are coupled and must be solved at once. In what follows we discuss the iterative procedure used to determine firstly the conditional fiscal limit distribution and then its unconditional, state-independent version.

Figure B.1 : Iteration scheme overview



The fully endogenous distribution of the fiscal limit and the associated default risk premium are determined iteratively (see Figure B.1). However, we introduce an efficient scheme that enables us to find these two unknowns iteratively (see Figure B.1). To simplify the notation in the following text we drop the explicit presence of the specific grid node (a_t, g_t, r_t, z_{t-1}) and denote $\mathscr{B} = \mathscr{B}(a_t, g_t, r_t, z_{t-1})$ the conditional distribution of the fiscal limit given the state (a_t, g_t, r_t, z_{t-1}). Thus, within the jth iteration we firstly calculate the jth approximation of the fiscal limit distribution $\mathscr{B}^{(j)}$ using the equations (A.1)–(A.8) while taking the previous $(j-1)$th iteration step approximations of the both default bond price $q^{(j-1)}$ and fiscal limit distribution $\mathscr{B}^{(j-1)}$ as given. Then, using the adjusted fiscal limit distribution $\mathscr{B}^{(j)}$ we solve numerically the nonlinear model prescribed by equations (A.9)–(A.13) to obtain the more precise judgement of the default bond price $q^{(j)}$. As illustrated on Figure B.1 above we repeat this procedure until the convergence (in the corresponding norms) of both the fiscal limit distribution and the default bond price is achieved, i.e.

$$\|\mathscr{B}^{(j)} - \mathscr{B}^{(j-1)}\|_{\mathscr{B}} < \varepsilon_{\mathscr{B}}, \qquad \text{and} \qquad \|q^{(j)} - q^{(j-1)}\|_q < \varepsilon_q ,$$

for some small[31] $\varepsilon_{\mathscr{B}}$ and ε_q. We refer to this procedure as to "main iteration procedure". Finally we derive the default risk premium.

Firstly, we need to discuss several technical aspects that must be taken into consideration when looking for the problem solution.

B.1 Projection, Interpolation and Grid Specification

To solve the problem we apply the well-known discrete state space technique. That is, we discretize the stochastic process for the technology, government consumption, level and regime

[31] The convergence requires the maximal inter-iteration pointwise change of both the fiscal limit distribution and the default price not to exceed 10^{-6}.

of transfers and (post- and pre-default) debt level and allow to choose the initial model state from a discrete set of points only. We solve the model using discrete state-space technique with different grid specifications and using the cubic spline interpolation (see e.g. C. de Boor (1978)). Employing cubic splines (instead of the usually preferred linear interpolation) enables us not only to describe the evolution of the control variables on the continuous space but also implies that trajectories satisfy desired analytical properties[32] and lead to a significantly smaller interpolation error than linear interpolation[33].

Proper grid specification has an essential impact on the exactness of the projection, interpolation and numerical integration. In order to match the properties of Slovak business cycle empirical distribution we prefer the non-uniformly distributed technology grid points to the evenly distributed points. Furthermore, the choice of such grid points (and their weights) arises from (inverse) Gauss-Legendre quadrature (see Abbott (2005)). On the other hand side we keep equidistant grid points to describe shocks in fiscal variables (government consumption and transfers). Our specification of grids has an impact on numerical integration methods used to solve this problem. Concretely, we rely on a combination of the Gauss-Legendre quadrature[34] and the 3/8 Simpson's rule[35] (see H. Jeffreys (1988)) when evaluating the multiple integral that is necessary to determine the risky bond price (A.12) and the associated default risk premium.

In the following text we fix the iteration step j and discuss in details how to obtain the jth approximation of the fiscal limit distribution and the risky bond price under the assumption that their $j-1$th approximations are known.

B.2 Fiscal Limit Distribution

The aim of the first part of the procedure is to update our judgement about the fiscal limit distribution and hence derive new $\mathscr{B}^{(j)}$ and the corresponding grid $\Delta^{(j)}$ of the actual default rate.

To obtain the distribution of the fiscal limit we employ the Markov-Chain Monte-Carlo simulation technique. Thus, on each grid point of the discretized model state-space

$$s_t = (a_t, g_t, r_t, z_{t-1}),$$

we randomly draw $N = 10^6$ series shocks for the productivity, government purchase, transfers and regime of transfers of length $T = 200$ (quarters) given their distribution functions. The generated series of shocks do not change between the iteration process. The specific length of

32 They are piecewise cubic, twice continuously differentiable and their second derivatives are zero at the end points. These interpolants are easier to evaluate than the high-degree polynomials used in standard polynomial interpolation. Furthermore, cubic splines do not exhibit oscillatory behaviour. We use the not-a-knot condition to determine the value of the derivatives at the end points.

33 In our model we apply cubic splines approach at each step j of the "main procedure" when projecting on the grids of the fiscal limit distribution $\mathscr{B}^{(j-1)}$ or bond price $q^{(j-1)}$. Obviously, after obtaining $\mathscr{B}^{(j)}$ or $q^{(j)}$ the corresponding spline is recalculated.

34 A Gaussian type of quadrature rule is typically more accurate than a Newton-Cotes (e.g. Simpson's rules) formulae, however it is computationally more complex.

35 We use 3/8 Simpson's rule rather than standard trapezoidal rule due to its higher accuracy (H. Jeffreys (1988)).

the shock series coincides with the length of the projections of age-related transfers given by Commission (2015).

Furthermore, to estimate correctly the stochastic default discount rate, we employ the default debt price $q_t = q(b_t^d, s_t)$ arising as the solution to the non-linear system (A.9)–(A.13) in the previous iteration[36]. Recalling Section 2.3.1, to obtain the distribution of the fiscal limit, at each step j of the "main procedure" one must solve the subsequent forward-looking implicit equation

$$b_{t-1}^{(j)} = \sum_{k=0}^{T} \rho_{t+k}^{(j-1)} \frac{\varsigma_{t+k}^{\max}}{1-\Delta_{t+k}^{(j-1)}}, \qquad \rho_{t+k}^{(j-1)} = \begin{cases} 1, & k=0, \\ \rho_{t+k-1}^{(j-1)} \dfrac{q_{t+k-1}^{(j-1)}}{1-\Delta_{t+k-1}^{(j-1)}}, & k>0, \end{cases} \tag{B.1}$$

at each point s_t of the discretized state-space and for any draws of shock series $\{\varepsilon_{t+k}^a\}_{k=1}^T$, $\{\varepsilon_{t+k}^g\}_{k=1}^T$, $\{\varepsilon_{t+k}^z\}_{k=1}^T$ and $\{\varepsilon_{t+k}^r\}_{k=1}^T$ and for

$$q_{t+k}^{(j-1)} = q_{t+k}^{(j-1)}\left((b_{t+k}^d)^{(j-1)}, s_{t+k}\right), \qquad \text{and} \qquad \Delta_{t+k}^{(j-1)} = \Delta_{t+k}^{(j-1)}\left(b_{t+k-1}^{(j-1)}, s_{t+k}\right), \tag{B.2}$$

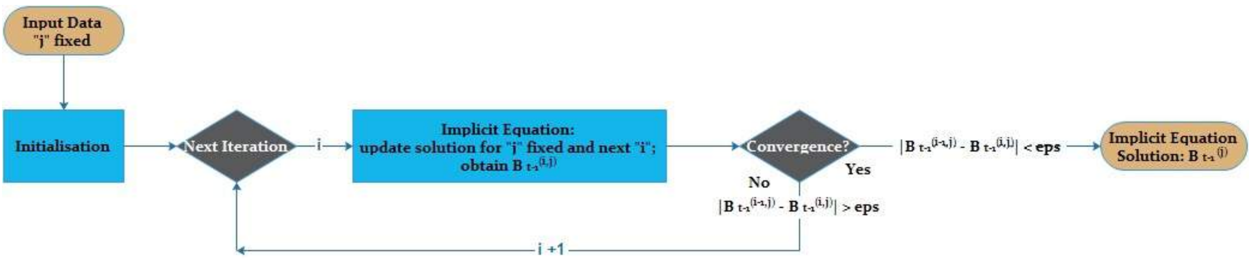
where s_{t+k} represents the time $t+k$ state and the time $t+k$ state

$$s_{t+k} = (a_{t+k}, g_{t+k}, r_{t+k}, z_{t+k-1}) .$$

It must be emphasized, that from the perspective of the "main" iteration procedure (illustrated on Figure B.1) both q and Δ from (B.2) are the $j-1$th approximations of the "true" q and Δ which are known (in terms of their prescription or corresponding grid values) at the beginning of the current, jth iteration of the "main" procedure. Furthermore, to avoid confusion, we use the index j to denote the iteration step of the main procedure (illustrated on Figure B.1), and within the "main" iteration j we distinguish between the time-step index k (for $k \in \{0, \ldots, T\}$) and i representing the fixed-point iteration step in the process of solving (B.1).

Next, in order to reduce the model complexity we moved from the stochastic maximal default rate $\delta_t \sim \Omega$ to its constant mean value $\delta = \mathbb{E}_\Omega[\delta_t]$. Therefore, the actual default rate Δ_t becomes the function of the fiscal limit distribution $\mathscr{B}$ only.

Figure B.2 : Fiscal Limit Distribution Phase: Fixed-point scheme overview



However, both the actual default rate Δ_t and bond price q_t depend on b_{t-1}, i.e. the present value of the sum of current & all future maximal surpluses (the right hand side of the equation (B.1)) is for the given grid point s_t and the associated draws of shock series the function of the time t begin-of-period debt b_{t-1} only (the left hand side of the equation (B.1)).

[36] This approach is perfectly consistent with the formula used to estimate the fiscal limit distribution as the expectation of future values of model random variables is conditioned by the time t information set.

Technically, one can easily deduce that highly nonlinear problem (B.1) is a contraction mapping. Thus, when solving (B.1) for a given initial state s_t and a set of shock time series $\{\varepsilon^a_{t+k}\}_{k=1}^T$, $\{\varepsilon^g_{t+k}\}_{k=1}^T$, $\{\varepsilon^z_{t+k}\}_{k=1}^T$ and $\{\varepsilon^r_{t+k}\}_{k=1}^T$ we apply the modified fixed-point iteration approach to determine the unknown begin-of-period debt b_{t-1} (see Figure B.2).

Concretely, for any $k \in \{0,\dots,T\}$ and (fixed) current main iteration procedure step j denote $q^{(j-1)}_{t+k}(b^{(i,j)}_{t+k})$ and $\Delta^{(j-1)}_{t+k}(b^{(i,j)}_{t+k-1})$ the time $t+k$ debt price and actual default rate, respectively, that were determined during the ith step of the fixed-point procedure (method of solving (B.3)) given the knowledge of the $j-1$th approximations of grids (functionals) q and Δ. In order to find the approximation of the unknown begin-of-period debt at the ith iteration, $b^{(i,j)}_{t-1}$ we need to solve the subsequent problem:

$$
\begin{aligned}
& b^{(i,j)}_{t-1}(s_t) - \sum_{k=0}^{T} \rho^{(j-1)}_{t+k}(b^{(i,j)}_{t+k-1}) \frac{\varsigma^{\max}_{t+k}(s_{t+k})}{1-\Delta^{(j-1)}_{t+k}(b^{(i,j)}_{t+k-1})} = 0\,, \\
& \rho^{(j-1)}_{t+k}(\cdot) = \begin{cases} 1\,, & k=0\,, \\ \rho^{(j-1)}_{t+k-1}(\cdot)\dfrac{q^{(j-1)}_{t+k-1}(\cdot)}{1-\Delta^{(j-1)}_{t+k-1}(\cdot)}\,, & k>0\,, \end{cases}
\end{aligned}
\tag{B.3}
$$

Concerning (B.3) there are two key points that must be emphasized. Firstly, we notice that the maximal primary surplus at time $t+k$ can be fully revealed directly using the randomly generated series of shocks up to time $t+k$ and initial state, since it depends only on the projected trajectories for productivity, government purchase, regime of transfers and transfers and it is independent of the initial guess of the debt. Therefore, since

$$
\varsigma^{\max}_{t+k} = \varsigma^{\max}_{t+k}(s_{t+k}) = \varsigma^{\max}_{t+k}(s_t;\{\varepsilon^a_{t+j}\}_{j=1}^k,\{\varepsilon^g_{t+j}\}_{j=1}^k,\{\varepsilon^r_{t+j}\}_{j=1}^k,\{\varepsilon^z_{t+j}\}_{j=1}^k)\,, \qquad k\in\{1,\dots,T\}\,.
$$

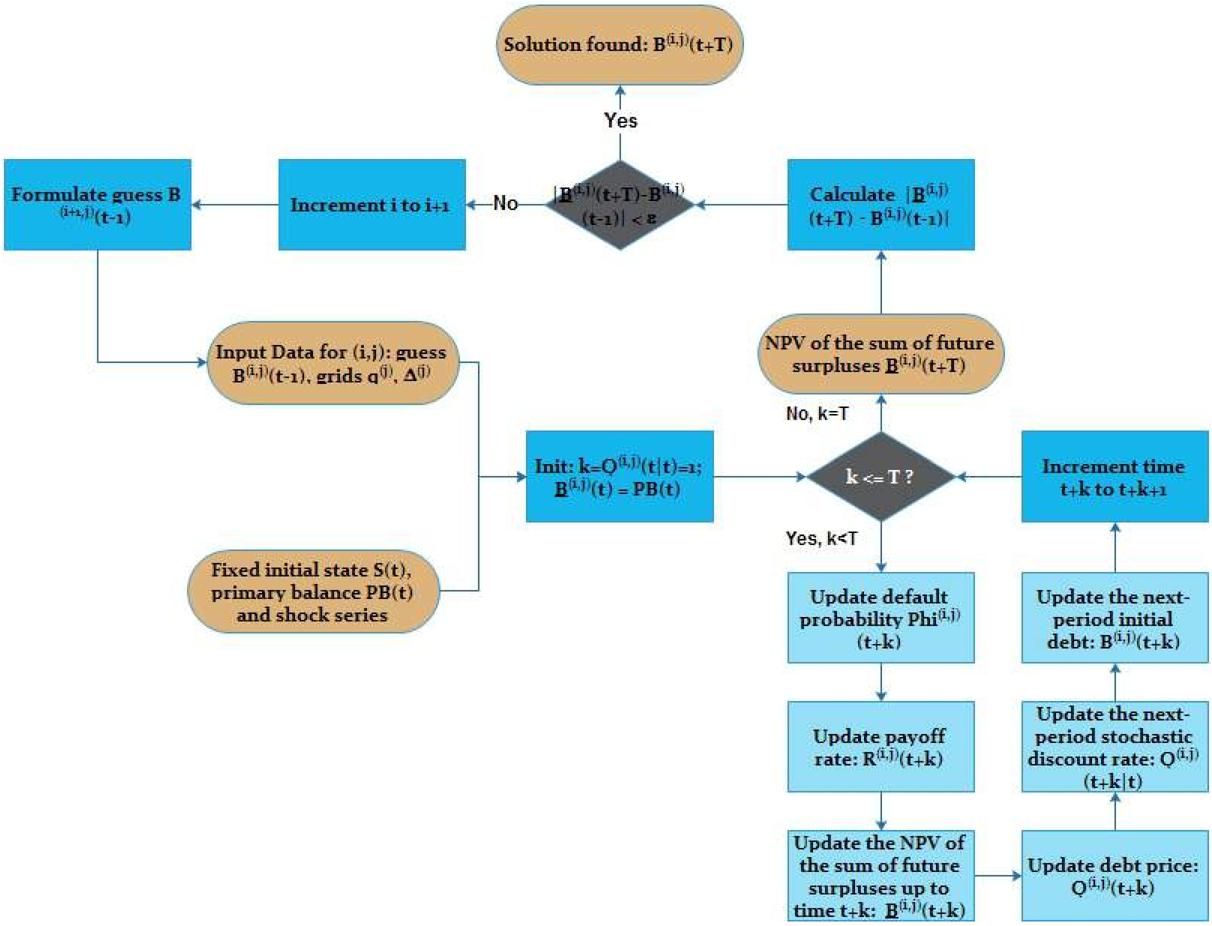
for a given initial state and the drawn of shocks, the value of $\varsigma^{\max}_{t+k}$ does not vary within the iterative procedure employed in solving the implicit equation (B.3).

Secondly, the $(i+1)$ approximation of the time $t+k-1$ begin-of-period debt $b^{(i+1)}_{t+k-1}$ depends on the unknown initial begin-of-period debt $b^{(i+1)}_{t-1}$, the known stream of maximal primary surpluses (which are independent of i) $\varsigma^{\max}_{t+j}$ and on the previous iteration approximations of the debt price and actual default rate functionals, $q^{(j-1)}$ and $\Delta^{(j-1)}$.

Hence the procedure of obtaining the solution to (B.3) during the jth loop of the "main procedure" i.e. the initial debt $b^{(j)}_{t-1}$ can be described as follows. Assume that the $(j-1)$th approximation of pricing and actual default rate grids, $q^{(j-1)}$ and $\Delta^{(j-1)}$ respectively, are given and during the jth loop of the "main" procedure we consider them for fixed. We initialise the stochastic discount rate $\rho^{(i,j)}_{t|t} = 1$ and using the initial state s_t and the set of shock time series $\{\varepsilon^a_{t+k}\}_{k=1}^T$, $\{\varepsilon^g_{t+k}\}_{k=1}^T$, $\{\varepsilon^r_{t+k}\}_{k=1}^T$ and $\{\varepsilon^z_{t+k}\}_{k=0}^T$ determine the whole paths of technology $\{a_{t+k}\}_{k=1}^T$, government consumption $\{g_{t+k}\}_{k=1}^T$, regime and level of transfers $\{r_{t+k}\}_{k=1}^T$, $\{z_{t+k}\}_{k=1}^T$ and primary surplus $\{\varsigma_{t+k}\}_{k=1}^T$. We set the right hand side of (B.3) to ς_t.

Then we proceed as follows. At the beginning of the ith step of the fixed-point iteration procedure applied when solving (B.3) we denote $\bar{b}^{(i,j)}_{t-1}$ the ith guess of the solution to (B.3) and set $b^{(i,j)}_{t-1} = \bar{b}^{(i,j)}_{t-1}$. Next, at each time-step $k \in \{0,\dots,T\}$ we firstly approximate the probability of default $\Phi^{(i,j)}_{t+k}$ by projecting the model state s_{t+k} and the pre-default debt $b^{(i,j)}_{t+k-1}$ on the grid $\Delta^{(j-1)}$.

Figure B.3 : Determining Fiscal Limit Distribution



This enables us to derive the actual payoff rate at time $t+k$, and the corresponding post-default debt $(b^d_{t+k})^{(i,j)}$ in the subsequent way:

$$(b^d_{t+k})^{(i,j)} = b^{(i,j)}_{t+k-1} r^{(i,j)}_{t+k}\,, \qquad r^{(i,j)}_{t+k} \equiv 1 - \delta\Phi^{(i,j)}_{t+k}\,.$$

Next, we update the sum of the net present value of discounted future primary balances inherited from previous periods by adding the net present value of the time $t+k$ discounted primary balance (independent of both i and j), $\rho^{(i,j)}_{t+k|t}\varsigma_{t+k}$ adjusted for the expected payoff rate, so

$$\widetilde{b}^{(i,j)}_{t+k|t} = \widetilde{b}^{(i,j)}_{t+k-1|t} + \frac{\rho^{(i,j)}_{t+k|t}}{r^{(i,j)}_{t+k}}\varsigma_{t+k}\,.$$

Note that using the stochastic discount rate $\rho^{(i,j)}_{t+k|t}$ at time $t+k$ prior its change is perfectly correct since $\rho_{t+k|t}$ is previsible, i.e.

$$\rho^{(i,j)}_{t+k|t} = \begin{cases} 1\,, & k=0\,,\\ \prod_{s=0}^{k-1} \frac{q^{(i,j)}_{t+s}}{r^{(i,j)}_{t+s}}\,, & k>0\,. \end{cases} \tag{B.4}$$

Then, we prepare variables for the next time-step: firstly, projecting the current state s_{t+k} and the post-default debt value $(b^d_{t+k})^{(i,j)}$ on the grid $q^{(j-1)}$ we recalculate the debt price $q^{(i,j)}_{t+k}$ and thanks to (B.4) one can directly adjust the next-period stochastic discount rate and using the government budget constraint determine the next-period initial (pre-default) debt:

$$\rho^{(i,j)}_{t+k+1|t} = \rho^{(i,j)}_{t+k|t} \frac{q^{(i,j)}_{t+k}}{r^{(i,j)}_{t+k}}, \qquad \text{and} \qquad b^{(i,j)}_{t+k} = \frac{(b^d_{t+k})^{(i,j)} - \varsigma_{t+k}}{q^{(i,j)}_{t+k}}.$$

We increment the time step counter and continue with this procedure until $k = T$. After the last time step we compare the accumulated net present value of all simulated discounted maximal primary surpluses on the time horizon $t, \ldots, t+T$ with the initial guess of the begin-of-period debt $\bar{b}^{(i,j)}_{t-1}$. The convergence of the fixed-point iteration scheme for the given initial state s_t, series of shocks and grids Δ^{j-1}, $q^{(j-1)}$ whenever

$$|\bar{b}^{(i,j)}_{t-1} - \widetilde{b}^{(i,j)}_{t+k|t}| < \varepsilon_b\,,$$

for some small ε_b[37]. Otherwise, we increment the iteration counter to $i+1$, update our guess $\bar{b}^{(i+1,j)}_{t-1}$ of the solution to (B.3) and repeat the procedure until the convergence is reached and so the point estimate at jth iteration of the main procedure of the fiscal limit for a given state and draw of shock series is obtained. Finally, by smoothing the aggregated simulated grid point estimates for all simulations we estimate the updated approximation of the fiscal limit distribution $\mathscr{B}^{(j)}$ and the corresponding grid $\Delta^{(j)}$ of the actual default rate (hence, the associated cubic splines must be recalculated). Furthermore, these two products are directly used within the second part of this main procedure iteration, i.e. in the process of adjusting the approximation of the debt pricing rule.

Initial Step Determination of the fiscal limit distribution is computationally expensive - however this is not true when we need to find the initial approximation of the distribution (i.e. when $j = 1$ in the "main procedure"). Indeed, since at the very beginning there is *no* prior information about the probability of country default available, we can simplify (A.8) and determine the distribution of the fiscal limit as follows:

$$\mathscr{B}^*_t \sim \mathbf{E}_t \sum_{k=0}^{\infty} \beta^k \frac{u^c_{t+k}}{u^c_t} \varsigma^{\max}_{t+k}\,, \qquad u^c_{t+j} = 1/c_{t+j}\,. \tag{B.5}$$

Evidently, the initial approximation of the fiscal limit distribution can be derived straightforwardly with no additional costs on projection & interpolation techniques and implicit schemes since all the relationships take the explicit forms.

How to form the guess? Finding the more exact approximation of the fiscal limit distribution implies the necessity of iterating the "main procedure" and hence solve the highly nonlinear

[37] Technically, we use the standard MATLAB method *fzero* (developed by T. Dekker and based on its FORTRAN version - see G. E. Forsythe and M. A. Malcolm and C. B. Moler (1976)) that uses a combination of bisection, secant, and inverse quadratic interpolation methods. Therefore, what we only need is to have a good initial judgement of the solution $\bar{b}^{(1,j)}_{t-1}$ for any j to speed-up the procedure.

implicit equation (B.3) for $j > 1$. Obviously a good initial guess $\bar{b}_{t-1}^{(1,j)}$ of the solution to (B.3) can speed-up this process significantly.

Therefore, to form our judgement about a possible solution to (B.3) at step $j > 1$ of the "main procedure" for a given initial state s_t and shock series draw we combine the information about the previous iteration solution at this grid point with the current solutions at neighbouring[38] points $x_{s_t} \in \mathcal{O}(s_t;\varepsilon)$:

$$\bar{b}_{t-1}^{(1,j)}(s_t;\varepsilon) = \omega_s b_{t-1}^{(j-1)}(s_t;\varepsilon) + (1-\omega_s) \sum_{x_{s_t} \in \mathcal{O}(s_t;\varepsilon)} \omega_{x_s} b_{t-1}^{(J)}(x_{s_t};\varepsilon)$$

where ε is a shorthand for a particular draw of shocks $\{\varepsilon^a_{t+k}\}_{k=1}^T, \{\varepsilon^g_{t+k}\}_{k=1}^T, \{\varepsilon^r_{t+k}\}_{k=1}^T, \{\varepsilon^z_{t+k}\}_{k=0}^T$ and ω_s, ω_{x_s} are some weights.

B.3 Default Interest Rate

Following the iterative procedure depicted on Figure B.1 within the second phase of each iteration j we need to solve the system of equations (A.10)–(A.12) given the approximation of the fiscal limit distribution $\mathscr{B}^{(j)}$ the output of the first phase of this iteration of the "main procedure". Thus, the debt pricing rule associated with the pre-default debt is determined by solving the nonlinear forward–looking model (A.10)–(A.12) assuming that tax rate follows a simple Taylor–like rule (A.9) and the actual default rate satisfies (A.13). Given the estimated fiscal limit distribution we solve this problem numerically employing the monotone mapping method. Intuitively, the system of the optimal condition (A.1)–(A.5), (A.9)–(A.13) along with the fiscal limit distribution (A.6)-(A.8) is converted into the set of the first order difference equations which are solved iteratively. Given the fixed point in the state–space and the initial guess of the debt rule, the aim is to find the final debt rule at that point, $b_t = f^b(\psi_t)$ which is the end-of-period, pre–default debt, function of the current state.

Hence, in each step j of the "main procedure" iteration we map the current state s_t, post-default debt b_t^d, default rate distribution $\delta_t \sim \Omega$ and the jth approximation of the fiscal limit distribution $\mathscr{B}^{(j)}$ to obtain the updated guess of the debt rule, i.e. the next-period pre-default debt

$$b_t^{(j)} = b_t^{(j)}(b_t^d, s_t, \Omega, \mathscr{B}^{(j)}) \equiv b_t^{(j)}(\psi_t^{(j)})$$

by solving the core equation of the model,

$$\frac{b_t^d + g_t + z_t(\psi_t^{(j)}) - \Theta_t(\psi_t^{(j)})}{b_t^{(j)}(\psi_t(j))} = \beta \mathbf{E}_t \left\{ \left[1 - \Delta_{t+1}^{(j)}(\psi_{t+1}^{(j)})\right] \frac{c_t(\psi_t^{(j)})}{c_{t+1}(\psi_{t+1}^{(j)})} \right\}. \quad \text{(B.6)}$$

Notice that the actual default rate grid $\Delta^{(j)}$ comes from the first phase of the jth iteration of the main procedure as the product of the given maximal default rate $\delta \sim \Omega$ (the distribution Ω is constant w.r.t. j) and the updated distribution of the fiscal limit, $\mathscr{B}^{(j)}$.

[38] Providing that the solution to (B.3) at some neighbouring point has not been derived yet we use the solution from the previous iteration – in that case we write $J = j - 1$, otherwise $J = j$.

The solution to the core equation above is determined numerically using the iterative algorithm of Sims[39]. Concretely, since (B.6) is an implicit forward-looking equation, for a given judgement of its solution $\hat{b}_t^{(l,j)}$ at the lth step of Sim's algorithm we firstly evaluate the left-hand side of (B.6),

$$\frac{b_t^d + g_t + z_t(s_t) - \Theta_t(b_t^d, s_t, \hat{b}_t^{(l,j)})}{\hat{b}^{(l,j)}}.$$

Then we continue with the evaluation of the multiple integral (i.e. the right-hand side of (B.6)),

$$\begin{aligned}\beta \mathbf{E}_t\left\{\left[1-\Delta_{t+1}^{(j)}(\psi_{t+1}^{(j)})\right]\frac{c_t(\psi_t^{(j)})}{c_{t+1}(\psi_{t+1}^{(j)})}\right\} = &\beta c_t(b_t^d, s_t, \hat{b}_t^{(l,j)}) \int_{\varepsilon_{t+1}^a}\int_{\varepsilon_{t+1}^g}\int_{\varepsilon_{t+1}^r}\int_{\varepsilon_{t+1}^z}\int_{\Omega}\int_{\hat{b}_t^{(l,j)}\sim\mathscr{B}^{(j)}} \frac{1-\Delta_{t+1}^{(j)}(\psi_{t+1}^{(l,j)})}{c_{t+1}(\psi_{t+1}^{(l,j)})}\\ =&\beta c_t(b_t^d, s_t, \hat{b}_t^{(l,j)}) \int_{\varepsilon_{t+1}^a}\int_{\varepsilon_{t+1}^g}\int_{\varepsilon_{t+1}^r}\int_{\varepsilon_{t+1}^z} \left[1-\Phi_{\mathscr{B}^{(j)}}(\hat{b}_t^{(l,j)}, s_{t+1})\right]\frac{1}{c_{t+1}(\psi_{t+1}^{(l,j)})|_{\text{no def.}}}\\ &+\beta c_t(b_t^d, s_t, \hat{b}_t^{(l,j)}) \int_{\varepsilon_{t+1}^a}\int_{\varepsilon_{t+1}^g}\int_{\varepsilon_{t+1}^r}\int_{\varepsilon_{t+1}^z} \Phi_{\mathscr{B}^{(j)}}(\hat{b}_t^{(l,j)}, s_{t+1}) \int_{\delta_{t+1}\sim\Omega}\frac{1-\delta_{t+1}}{c_{t+1}(\psi_{t+1}^{(l,j)})|_{\text{def.}}}.\end{aligned}$$

The integral above is evaluated numerically using Gauss-Legendre quadrature (along the technology grid), 3/8 Simpson's rule and the trapezoidal rule (maximal default rate grid Ω). Note that the next-period private consumption c_{t+1} is obtained using (A.1)–(A.3) and (A.9)–(A.11) while the future actual default $\Delta_{t+1}^{(j)}$ is approximated using projection and interpolation techniques[40].

The convergence in Sim's algorithm is achieved when the difference between the left and right side of the equation is small enough (we require less than 10^{-6}).

After obtaining the decision rule (next-period pre-default debt $b_t^{(j)}$) for a given initial state s_t and post-default debt b_t^d in the discrete state–space, we employ the budget constraint (A.10) to find the approximation of the bond–pricing rule, $q_t^{(j)}$ directly as

$$q_t^{(j)} = \frac{b_t^d + g_t + z_t(\psi_t^{(j)}) - \Theta_t(\psi_t^{(j)})}{b_t^{(j)}(\psi_t(j))}. \tag{B.7}$$

Finally, as mentioned earlier, providing that in the end of the jth iteration of the "main procedure" both the fiscal limit distribution and pricing rule converge (i.e. the differences between the distributions and rules obtained in the current and past iterations are sufficiently small) the iterative algorithm terminates and we evaluate the default risk premium. Otherwise, the last approximation of the bond pricing rule is passed to the first phase of the subsequent iteration $j+1$ of the "main procedure".

[39] Unfortunately, there is no guarantee that this algorithm is able to find the solution to (B.6) for every possible parametrization such that it lives within some reasonable boundaries. Therefore, the model is extended to include some built–in approximation techniques.

[40] To obtain the probability of default for a given debt guess $\hat{b}_t^{l,j}$, $\Phi_{\mathscr{B}^{(j)}}(\hat{b}_t^{l,j})$ we project the next-period system state s_{t+1} and the debt guess $\hat{b}_t^{l,j}$ onto the jth iteration approximation of the fiscal limit distribution grid $\mathscr{B}^{(j)}$ and use the cubic splines (associated with the jth iteration of the main procedure) we smoothly approximate value of the country's default probability.

B.4 Unconditional Distribution of Fiscal Limits

The unconditional distribution of the fiscal limit $\widetilde{\mathscr{B}}$ can be obtained in a similar way as the state-dependent distribution $\mathscr{B} = \mathscr{B}(a_t, g_t, r_t, z_{t-1})$. However, several differences between these two distributions must be highlighted.

Firstly, for each simulation $i \in \{1, \ldots, N = 10^6\}$ we randomly draw the shocks for productivity, government purchase and regime of transfers of length $T_{ini} + T_H$ (quarters) given their distribution functions, calculate the resulting levels of technology and government purchase and drop the first T_{ini} observations.[41] Next, due to expansive nature of transfers we assume that during this initial period they follow an autoregressive process[42]. Therefore, to find the unconditional distribution we let each simulation i to start from the point

$$\widetilde{s}^{(i)} = (a_{T_{ini}}, g_{T_{ini}}, r_{T_{ini}}, z_{T_{ini}-1}).$$

Next, unconditional distribution implies dimension reduction when projecting on the default rate grid Δ. Indeed, as Δ becomes independent of the state s, we map a given value on a certain begin-of-period debt. Lower complexity of the problem causes significant time-savings in both phases of the iterative procedure.

Furthermore, to study the gradual adjustment and stabilisation of (originally exponentially growing) transfers we extend the set of possible regimes by the stabilising state r^0. Therefore, it holds that

$$z_t(r_t) = \begin{cases} z_{t-1} + \varepsilon_t^z, & r_t = 0, \\ \mu_t^{(1)} z_{t-1} + \varepsilon_t^z, & r_t = 1, \\ \mu_t^{(2)} z_{t-1} + \varepsilon_t^z, & r_t = 2. \end{cases} \tag{B.8}$$

Next, since we assume that as time goes by policy-makers are more likely to keep transfers stable, we introduce the time-varying transition matrix M_t and calibrate it such that the idea of gradual preference of stable regime of are-related expenses is captured:

$$M_t = \begin{bmatrix} 1 - p_{01}(t) - p_{02}(t) & p_{01}(t) & p_{02}(t) \\ p_{10}(t) & 1 - p_{10}(t) - p_{12} & p_{12} \\ p_{20}(t) & p_{21} & 1 - p_{20}(t) - p_{21} \end{bmatrix}, \tag{B.9}$$

where the time-dependent matrix elements $p_{01}(t)$, $p_{02}(t)$, $p_{10}(t)$ and $p_{20}(t)$ satisfy

$$p_{0i}(t) = p_{0i}^{(\infty)} - \left[p_{0i}^{(\infty)} - p_{0i}^{(0)}\right] e^{-\theta_{0i} t}, \qquad p_{j0}(t) = p_{j0}^{(\infty)} - \left[p_{j0}^{(\infty)} - p_{j0}^{(0)}\right] e^{-\theta_{j0} t}, \qquad i, j \in 1, 2,$$

and θ_{0i} and θ_{j0} specify the transfers adjustment speed. Next, $M^{(0)}$ and $M^{(\infty)}$ are constant initial and terminal transitory matrices given as follows:

$$M^{(k)} = \begin{bmatrix} 1 - p_{01}^{(k)} - p_{02}^{(k)} & p_{01}^{(k)} & p_{02}^{(k)} \\ p_{10}^{(k)} & 1 - p_{10}^{(k)} - p_{12} & p_{12} \\ p_{20}^{(k)} & p_{21} & 1 - p_{20}^{(k)} - p_{21} \end{bmatrix}, \qquad k \in \{0, \infty\}.$$

[41] In this study we set $T_{ini} = T_H = 200$. However, since transfers are explosive the value of T_H is determined by the length of their projections (the distribution shifts to the right as T_H increases). To allow a randomly large simulation horizon T_H we need to introduce a kind of a resolution scheme (or stable regime of transfers with increasing probability of following it) that would prevent the transfers from blowing-up and so would guarantee the distribution convergence over time.

[42] Standard deviation and autocorrelation parameters of the autoregressive process of transfers are estimated from historical Slovak quarterly data observed between 2000 and 2016.

The algorithm works as subsequently. Firstly, for each simulation $i \in \{1, \ldots, N = 10^6\}$ we randomly draw the shocks for productivity, government purchase and regime of transfers of length $T_{ini} + T_0 + T_H$ (quarters) given their distribution functions, calculate the resulting levels of technology and government purchase and drop the first T_{ini} observations. While during the first T_{ini} periods we assume that transfers follow an autoregressive process, later on we let them to switch between three regimes and evolve accordingly the time-varying transition matrix $M_{t-T_{ini}}$. Then, to find the (time T_{ini}) expected $T_{ini} + T_0$ unconditional distribution of the fiscal limits we let each simulation i to start from the point

$$\widehat{s}_{T_0}^{(i)} = \left(a_{T_{ini}+T_0}, g_{T_{ini}+T_0}, r_{T_{ini}+T_0}, z_{T_{ini}+T_0-1}\right) .$$

The distribution is calculated assuming the time horizon T_H.

Appendix C Calibration and Grid Specification

C.1 Maximal Tax Revenues

The household choices about their level of consumption and labour supply only depend on the income tax rate τ_t and the exogenous state variables , technology a_t and government purchase g_t.
Assume the utility function is $u(c,h) = \log c + \phi \log(1-h)$.

Optimal Tax Rate: The household first-order conditions (see (11a)) can be written as

$$c_t = \frac{(a_t - g_t)(1-\tau_t)}{1+\phi-\tau_t}, \tag{C.1}$$

$$h_t = \frac{a_t(1-\tau_t)+\phi g_t}{a_t(1+\phi-\tau_t)}. \tag{C.2}$$

Then, the first derivative of the tax revenue $\Theta_t = \tau_t a_t h_t$ with respect to the tax rate τ_t,

$$\frac{\partial \Theta_t}{\partial \tau_t} = \frac{a_t \tau_t^2 - 2a_t(1+\phi)\tau_t + (1+\phi)(a_t+\phi g_t)}{(1+\phi-\tau_t)^2}, \tag{C.3}$$

have two distinct roots[43]

$$0 < \tau_t^{(-)} \equiv 1+\phi - \sqrt{\phi(1+\phi)\frac{a_t-g_t}{a_t}} < 1 < \tau_t^{(+)} \equiv 1+\phi+\sqrt{\phi(1+\phi)\frac{a_t-g_t}{a_t}}. \tag{C.4}$$

Thus, since $\partial\Theta_t/\partial\tau_t < 0$ iff $\tau_t \in (\tau_t^{(-)}, \tau_t^{(+)})$ one can straightforwardly deduce that

$$\tau_t^{\max} \equiv \tau_t^{(-)} = 1+\phi-\sqrt{\phi(1+\phi)\frac{a_t-g_t}{a_t}} \tag{C.5}$$

is the unique tax revenues maximiser and

$$\Theta_t^{\max} \equiv \Theta_t^{\max}(a_t, g_t) = (1+2\phi)a_t - \phi g_t - 2\sqrt{\phi(1+\phi)a_t(a_t-g_t)}. \tag{C.6}$$

Inasmuch as

$$\frac{\partial \Theta_t^{\max}}{\partial g_t} = -\phi + \sqrt{\frac{\phi(1+\phi a_t)}{a_t-g_t}} > 0, \qquad \phi > 0, a_t > g_t > 0,$$

the maximal tax revenues $\Theta_t^{\max}$ increases with the level of government purchase, g_t. Next, as

$$\frac{\partial \Theta_t^{\max}}{\partial a_t} = 1+2\phi-\phi\zeta_g - (2a_t - g_t - \zeta_g a_t)\sqrt{\frac{\phi(1+\phi)}{a_t(a_t-g_t)}} \in (0,1)$$

for any ζ_g small enough (any any negative), $\Theta_t^{\max}$ increases with the technology, a_t and moreover, fluctuations in exogenous productivity are projected into changes in the maximal tax revenues with lower magnitude[44].

[43] Since $a_t > g_t > 0$ and $\phi > 0$, it is evident that $0 < \tau_t^{(-)} < 1 < \tau_t^{(+)}$.
[44] This attribute of the maximal tax revenues is not in consistence with our observation of Slovak data.

Finally, combining (C.1)–(C.2) with (C.5) leads to the following optimal levels of consumption and labour supply depending on current technology and government purchase assuming that the tax rate is set such that it maximises the tax revenues:

$$h_t = 1 - \sqrt{\frac{\phi}{1+\phi}\left(1 - \frac{g_t}{a_t}\right)}, \tag{C.7}$$

$$c_t = -\sqrt{\phi \frac{a_t(a_t - g_t)}{1+\phi}} + (a_t - g_t)\,. \tag{C.8}$$

Labour supply declines with technology but increases with government purchase. The opposite behaviour is typical for consumption – it grows with technology, but falls with government purchase.

C.2 Calibration

In order to calibrate the model properly and determine the coefficient ϕ we assume that g/y, z/y, b/y and β (on annual frequency) are given and technology $a = 1$, so the steady state tax rate satisfies

$$\tau = \frac{b}{y}(1-\beta) + \left(\frac{z}{y} + \frac{g}{y}\right) \tag{C.9}$$

Next, we set the equilibrium labour supply $h = 0.25$, so $y = 0.25$. Then, plugging (C.9) into (C.1) evaluated in the steady–state leads to the following:

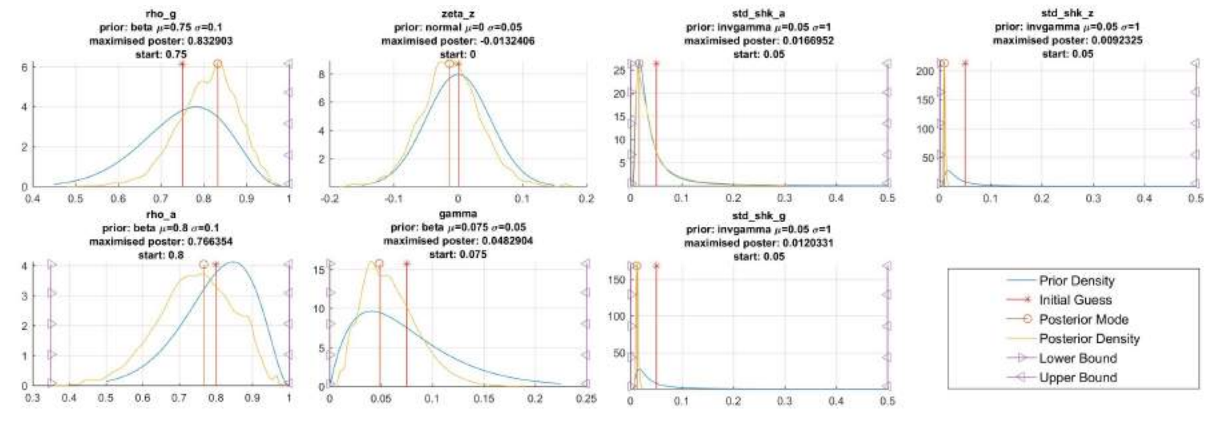
$$\phi = (1-\tau)\left(\frac{1}{h} - 1\right)\left[1 - \frac{g}{y}\right]^{-1}. \tag{C.10}$$

To derive the distribution of the fiscal limit we generate 10^4 sets of shock series $\{\varepsilon^a_{t+k}\}^T_{k=1}$, $\{\varepsilon^g_{t+k}\}^T_{k=1}$, $\{\varepsilon^z_{t+k}\}^T_{k=1}$, and $\{\varepsilon^r_{t+k}\}^T_{k=1}$ each of length $T = 50$ which remain unchanged during the main procedure. The problem convergence requires that in the main procedure the maximal inter-iteration point-wise change of both the fiscal limit distribution and the default price do not exceed 10^{-6}, i.e. we set $\varepsilon_{\mathscr{B}} = \varepsilon_q = 10^{-6}$. Similarly, at each step j of the main procedure implicit problems that must be solved in order to determine j-approximation of the fiscal limit distribution (B.3) and the debt rule (and thus the debt price) (B.6) have the convergence errors set to 10^{-9}, i.e. $\varepsilon_b = \varepsilon_r = 10^{-9}$. In higher iterations of the main procedure we form the guess of the begin-of-period debt in the implicit problem (B.3) by combining with the equal weights the previous iteration solution with the already determined current (or past) solutions at neighbouring points[45].

C.3 Business Cycle in Slovakia

[45] Notice that in case of the initial state s_t that is the *inner* point of the state grid (except of the regime of transfers dimension r), we set $\omega_s = .5$, $\omega_{x_{a-}} = \omega_{x_{a+}} = 1/8$, $\omega_{x_{g-}} = \omega_{x_{g+}} = 1/8$, $\omega_{x_{z-}} = \omega_{x_{z+}} = 1/8$ and $\omega_{x_{r-}} = \omega_{x_{r+}} = 1/8$. Here we assume that $\omega_{x_{a-}}$, $\omega_{x_{g-}}$, and $\omega_{x_{z-}}$ are weights of the solution of neighbouring points determined already during this iteration while $\omega_{x_{a+}}$, $\omega_{x_{g+}}$, and $\omega_{x_{z+}}$ come from the previous iteration of the main procedure.

Figure C.1 : Bayesian Estimation



Model Dynamics: Estimation Priors and Posteriors

Table C.1 : Bayesian Estimation

Parameter		Prior Type	Prior Mean	Prior St.Dev.	Posterior Mean	Posterior St.Dev	Confidence
Technology persistence	ρ_a	β	0.80	0.1	0.7664	0.0743	[0.6009, 0.9136]
Government purchase persistence	ρ_g	β	0.75	0.1	0.8329	0.0702	[0.7029, 0.9274]
Transfers sensitivity to business cycle	ζ_g	$\mathcal{N}$	0	0.1	-0.0132	0.0439	[-0.082, 0.0574]
Tax sensitivity to debt deviation	γ	β	0.075	0.05	0.0483	0.0240	[0.0180, 0.0947]
Technology shock volatility	σ_a	Γ^{-1}	0.05	0.01	0.0167	0.0561	[0.0062, 0.0802]
Government purchase shock volatility	σ_g	Γ^{-1}	0.05	0.01	0.0120	0.0025	[0.0093, 0.0174]
Transfers shock volatility	σ_z	Γ^{-1}	0.05	0.01	0.0092	0.0018	[0.0071, 0.0133]

Model priors, posteriors and 90% confidence intervals. Data source: Eurostat, National Bank of Slovakia

Table C.2 : Bayesian Estimation

Parameter		Prior Type	Prior Mean	Prior St.Dev.	Posterior Mean	Posterior St.Dev	Confidence
Technology persistence	ρ_a	β	0.80	0.1	0.7664	0.0743	[0.6009, 0.9136]
Government purchase persistence	ρ_g	β	0.75	0.1	0.8329	0.0702	[0.7029, 0.9274]
Transfers sensitivity to business cycle	ζ_g	$\mathcal{N}$	0	0.1	-0.0132	0.0439	[-0.082, 0.0574]
Tax sensitivity to debt deviation	γ	β	0.075	0.05	0.0483	0.0240	[0.0180, 0.0947]
Technology shock volatility	σ_a	Γ^{-1}	0.05	0.01	0.0167	0.0561	[0.0062, 0.0802]
Government purchase shock volatility	σ_g	Γ^{-1}	0.05	0.01	0.0120	0.0025	[0.0093, 0.0174]
Transfers shock volatility	σ_z	Γ^{-1}	0.05	0.01	0.0092	0.0018	[0.0071, 0.0133]

Model priors, posteriors and 90% confidence intervals. Data source: Eurostat, National Bank of Slovakia

Table C.3 : Descriptive Statistics for Slovak output gap data

	Range	Mean	Standard Deviation	Interquartile Range	α-Quantiles (0, 0.05, 0.15, 0.85, 0.95, 1)	Skewness	Kurtosis
Annual	17.8244	-0.1986	2.0241	1.5194	-9.4710 -3.1601 -1.6065 0.9315 3.7674 8.3534	0.8914	8.3644
Quarterly	19.1882	-0.2010	1.9514	1.5133	-9.4710 -3.1574 -1.6061 0.8865 3.6702 8.3534	1.0878	8.2908

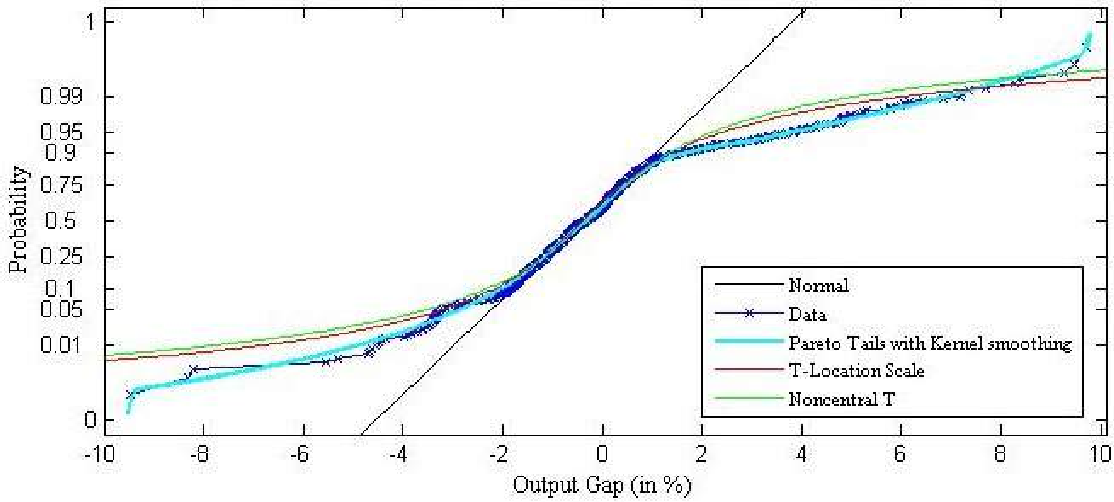
Source: Slovak MinFin, NBS, EC, BIS (2000-2014)

Table C.4 : Slovak output gap & empirical distribution

	Lower Tail		Upper Tail		Interior	Neg. Log-Likelihood
	Range	Distribution	Range	Distribution		
Annual	$x < -1.6787$ $\alpha < 0.15$	Generalised Pareto (0.1375,1.1532)	$x > 1.0552$ $\alpha > 0.85$	Generalised Pareto (0.1060, 2.8302)	Interp. kernel smooth cdf	403.6468
Quarterly	$x < -1.6464$ $\alpha < 0.15$	Generalised Pareto (0.1475,0.9856)	$x > 2.5024$ $\alpha > 0.925$	Generalised Pareto (0.1336, 3.0415)	Interp. kernel smooth cdf	1.6195e+03

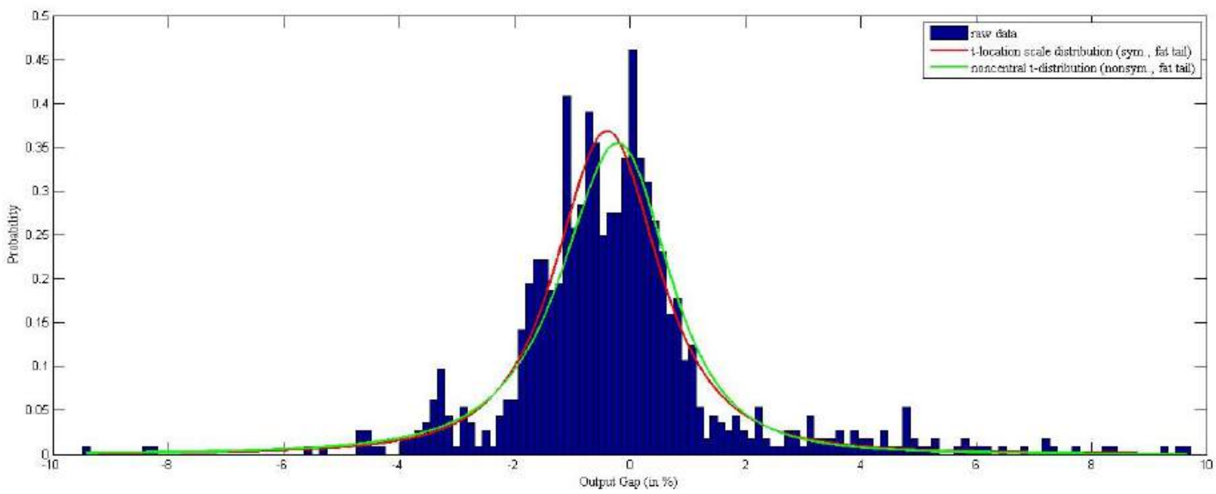
Characteristics of the empirical heavy-tailed distribution estimate fitting the Slovak output gap data

Figure C.2 : Non-normality of output gap in Slovakia



Q-Q plot for the comparison of the output gap data distribution (blue line with markers) to the normal distribution (black dashed line) empirical Pareto-tailed kernel smoothing distribution (thick cyan line) and location-scale T-distribution (thin red line).

Figure C.3 : Probability distribution function of the Slovak business cycle

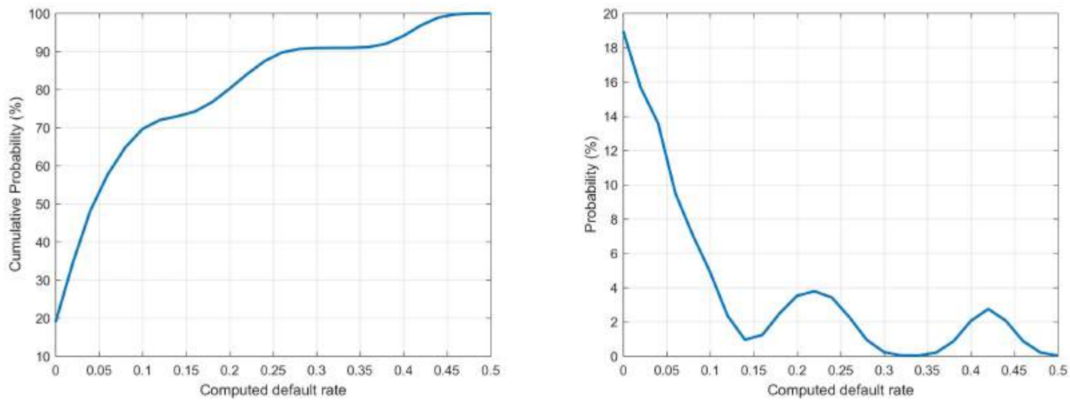


Estimation of the Slovak business cycle PDF based on data from 2000-2014. We use the t-location scale distribution (with parameters obtained from the maximal likelihood procedure) to fit the data properly.

C.4 Stochastic Default Rate

At time t the stochastic default rate δ follows the empirical distribution Ω computed by Bi and Leeper (2010) and Bi (2012) from the sovereign debt defaults and restructures observed in the emerging market economies during the period of 1983 to 2005.

Figure C.4 : Stochastic Default Rate Distribution



Cumulative and Probability Distribution Functions

C.5 Specification of Grids

Within this model we distinguish between grids used to approximate shocks and those employed to describe variable levels. We employ cubic splines when interpolating over the jth step-specific default rate distribution grid $\Delta^{(j)}$ and price grid $q^{(j)}$.

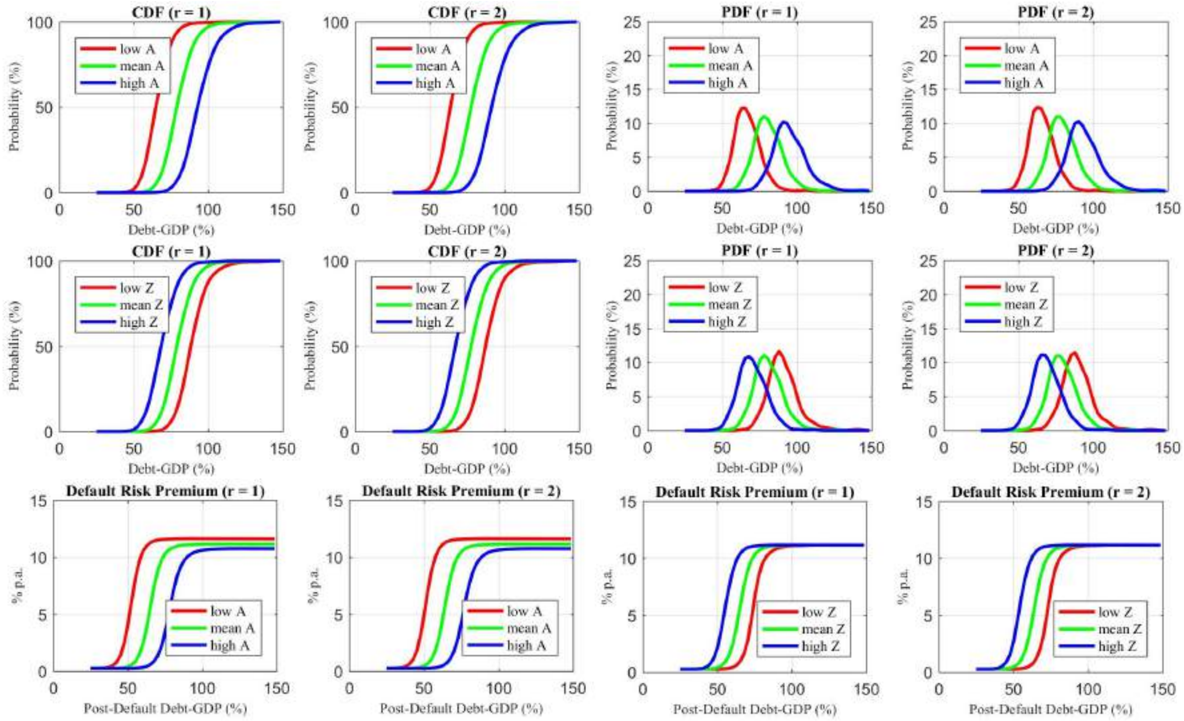
Table C.5 : Specification of grids

Variable		Integration Method	Grid Points		Grid Extrema	
			Shocks	Levels	Shocks	Levels
technology	a	Gauss-Legendre Quadrature	31	11	$\pm 4\sigma_a$	$a\exp\{\pm 4\sigma_a\}$
government purchase	g	3/8 Simpson's Rule	19	11	$\pm 3\sigma_g$	$g\exp\{\pm 3\sigma_g\}$
level of transfers	z	3/8 Simpson's Rule	19	11	$\pm 3\sigma_z$	$z(1 \pm 3\sigma_z)$
regime of transfers	r	-	2	2	-	$\{r_1, r_2\}$
empirical default rate	δ	Trapezoidal Rule	-	26	-	$\{0, 0.5\}$

Newton-Cotes rules are applied on grids with equidistant points.

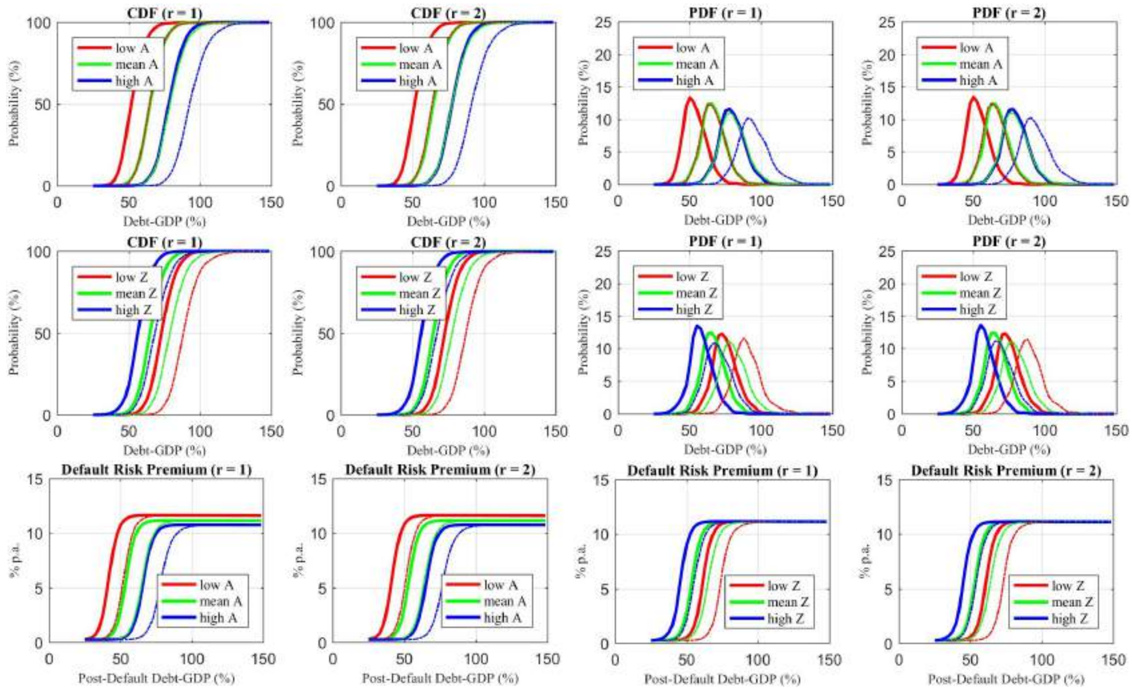
Appendix D Results

Figure D.1 : Baseline Scenario



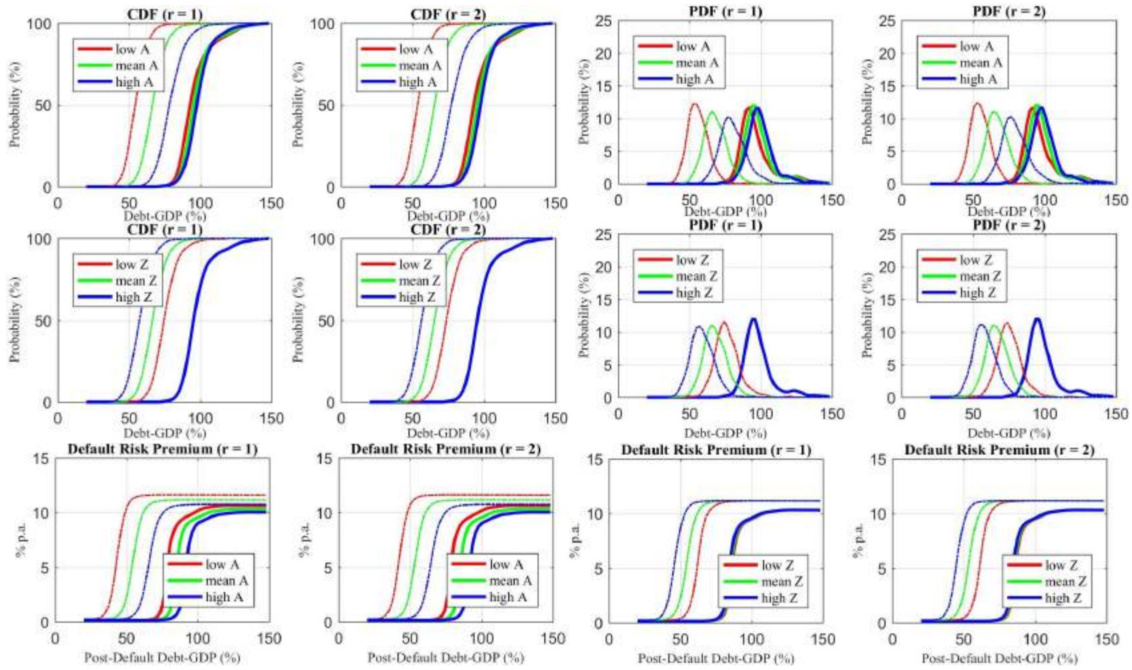
Probability and cumulative distribution functions of the fiscal limit and the corresponding default risk premium.

Figure D.2 : Higher Growth Rate of Transfers



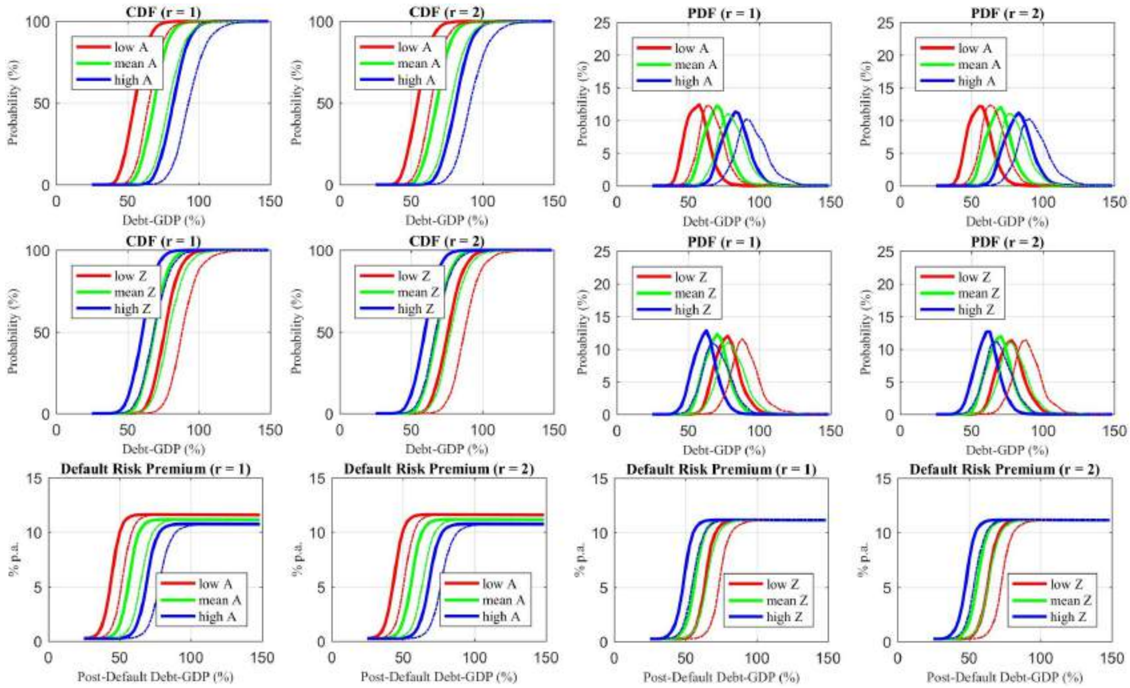
PDF and CDF of the fiscal limit and the corresponding default risk premium, growth rate of transfers (in both regimes) is higher such that the overall increase in transfers/GDP is by 10% higher w.r.t.the baseline scenario Dashed lines correspond to the baseline scenario.

Figure D.3 : Lower Growth Rate of Transfers



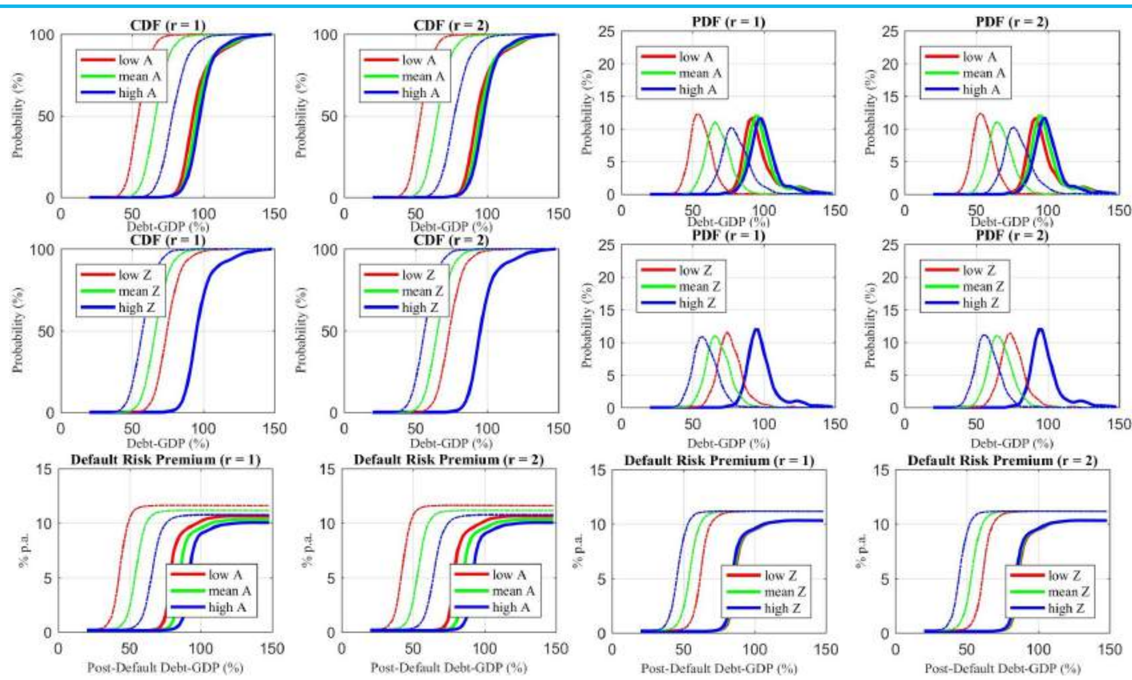
PDF and CDF of the fiscal limit and the corresponding default risk premium, growth rate of transfers (in both regimes) is higher such that the overall decrease in transfers/GDP is by 10% higher w.r.t.the baseline scenario Dashed lines correspond to the baseline scenario.

Figure D.4 : More Volatile Transfers



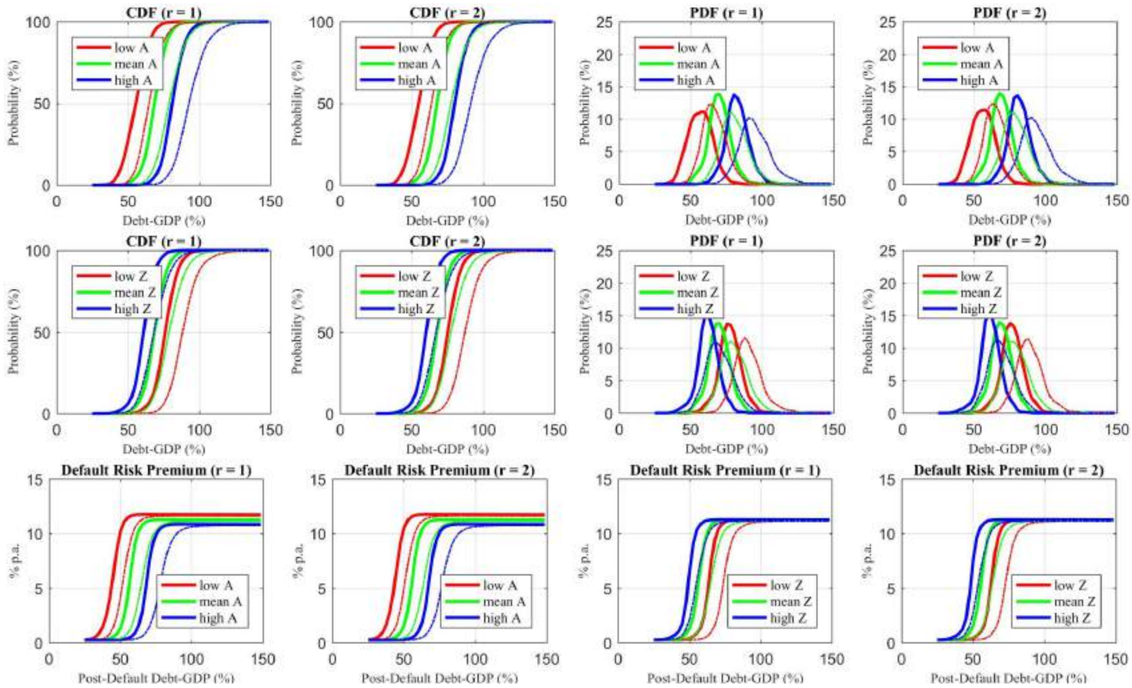
PDF and CDF of the fiscal limit and the corresponding default risk premium, standard deviation of shocks to transfers is higher by 20% w.r.t.the baseline scenario. Dashed lines correspond to the baseline scenario.

Figure D.5 : More Credible Fiscal Policy



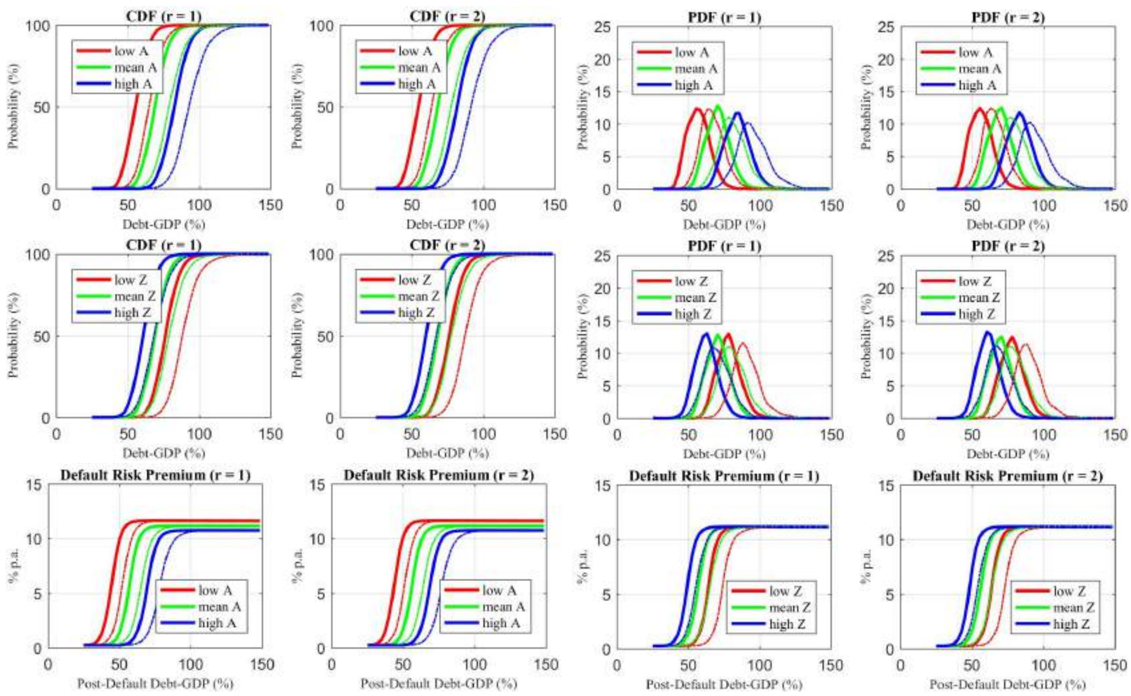
PDF and CDF of the fiscal limit and the corresponding default risk premium, transfers reside in the NPC regime twice longer in compare to the baseline scenario. Dashed lines correspond to the baseline scenario.

Figure D.6 : Higher Risk-free Interest Rate



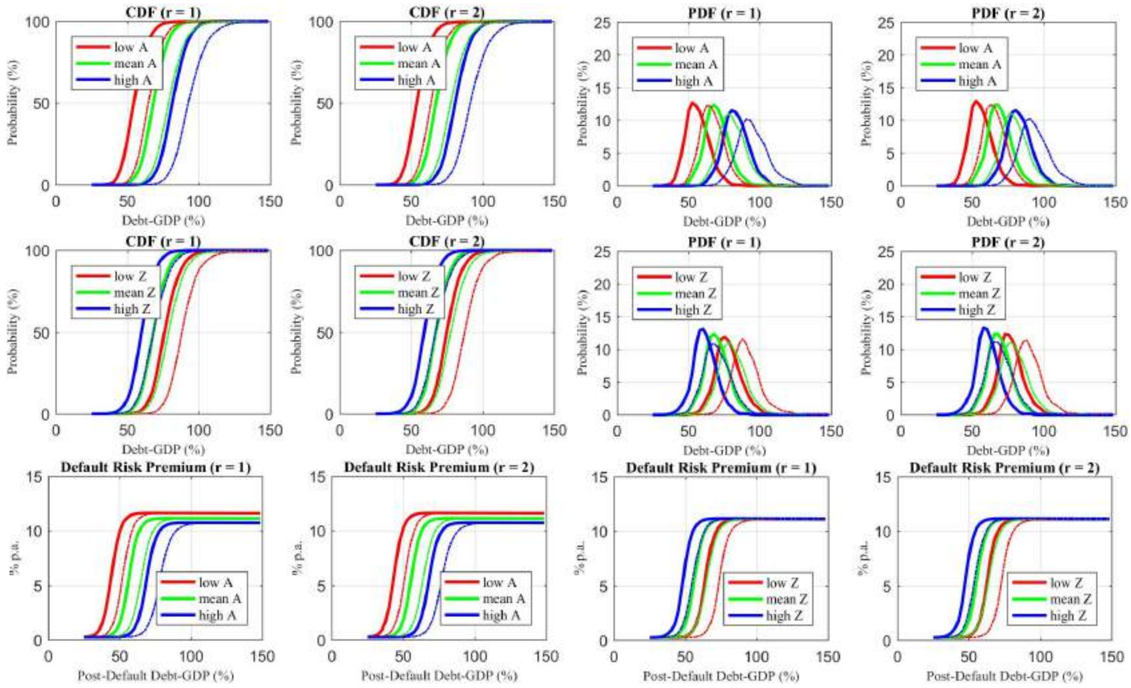
PDF and CDF of the fiscal limit and the corresponding default risk premium, risk-free interest rate is higher by 1 p.p. w.r.t.the baseline scenario. Dashed lines correspond to the baseline scenario.

Figure D.7 : More Volatile Technology



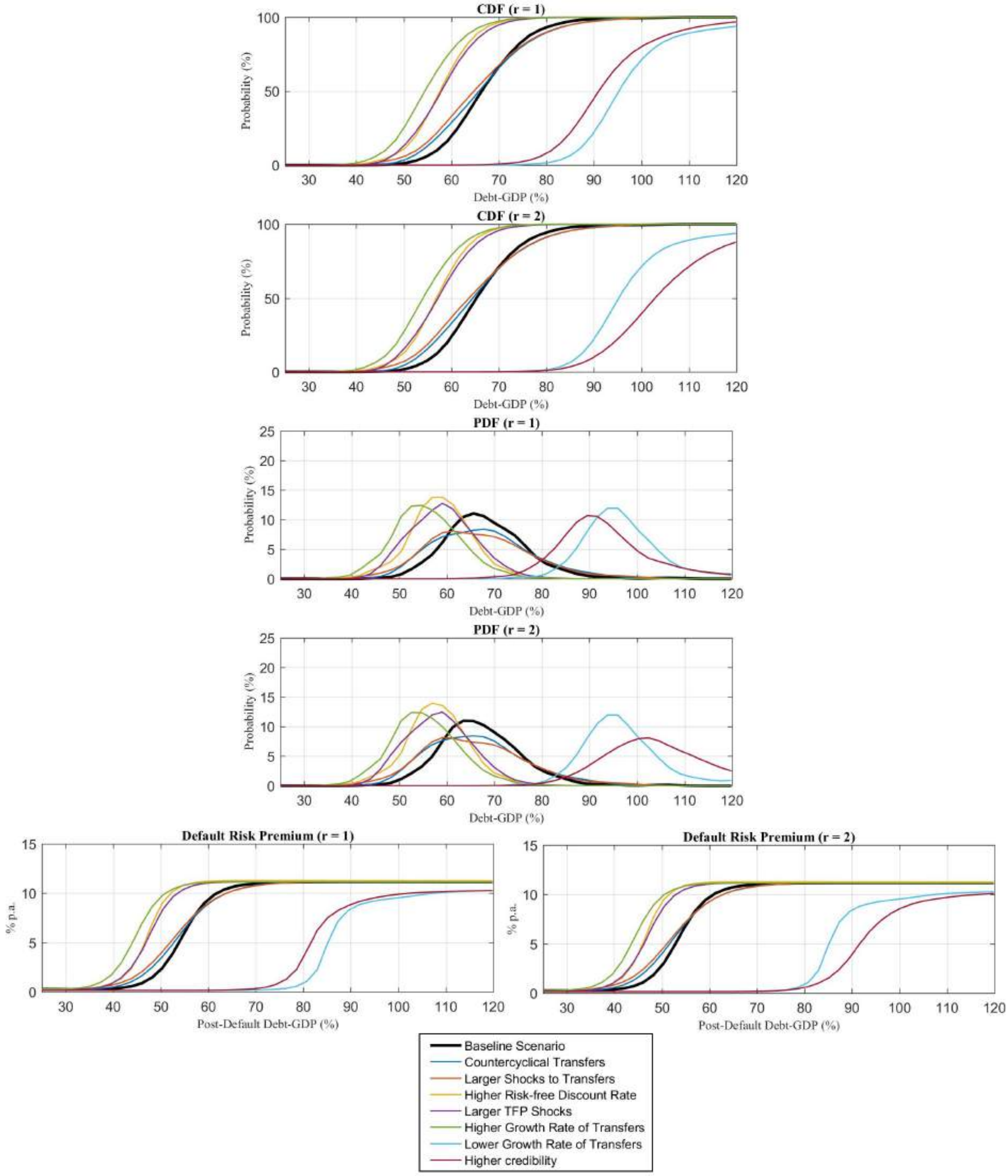
PDF and CDF of the fiscal limit and the corresponding default risk premium, standard deviation of shocks to technology is higher by 20% w.r.t.the baseline scenario. Dashed lines correspond to the baseline scenario.

Figure D.8 : More Responsive Tax



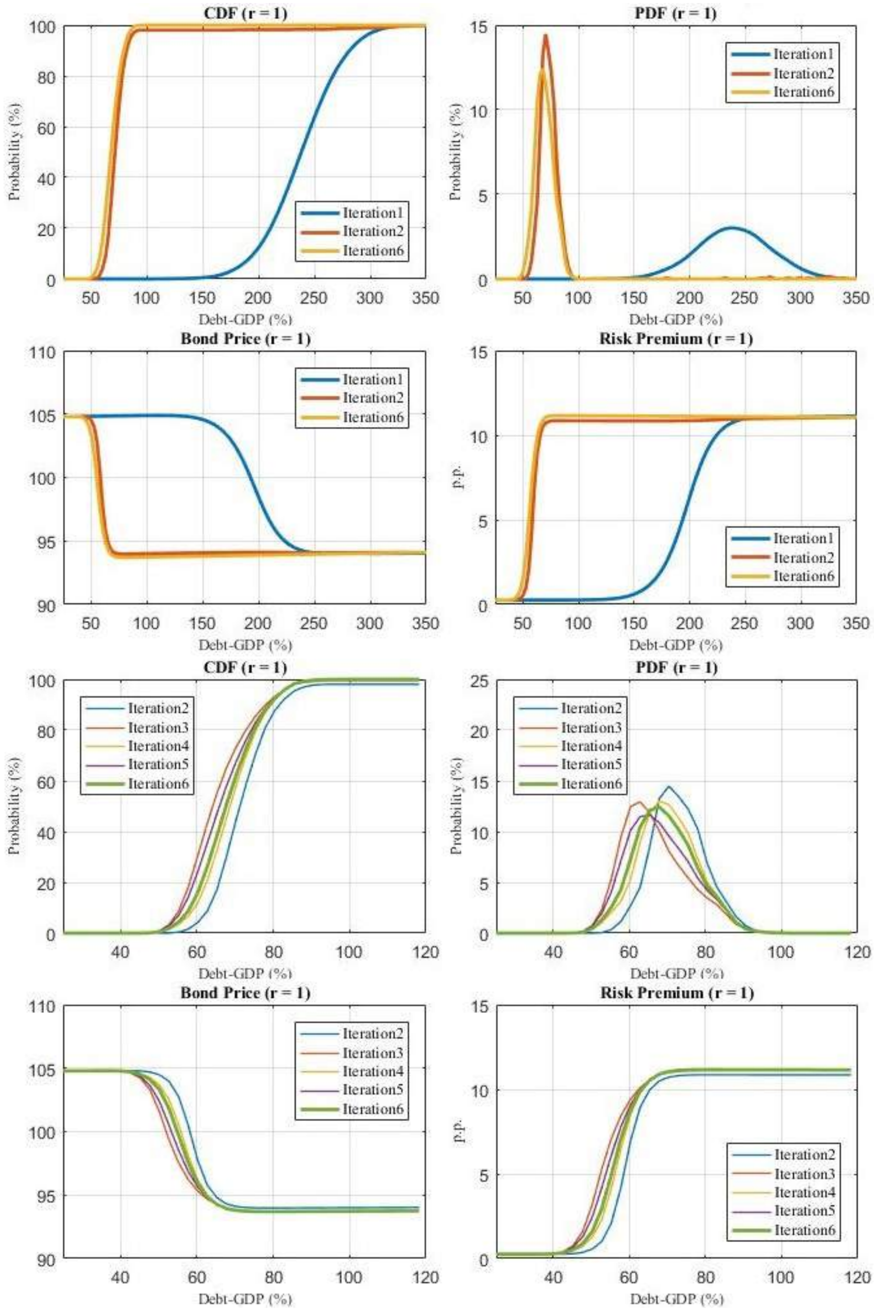
PDF and CDF of the fiscal limit and the corresponding default risk premium, tax rate responsiveness to debt deviation is higher by 20% w.r.t.the baseline scenario. Dashed lines correspond to the baseline scenario.

Figure D.9 : Sensitivity analysis: Fiscal limit and risk premium



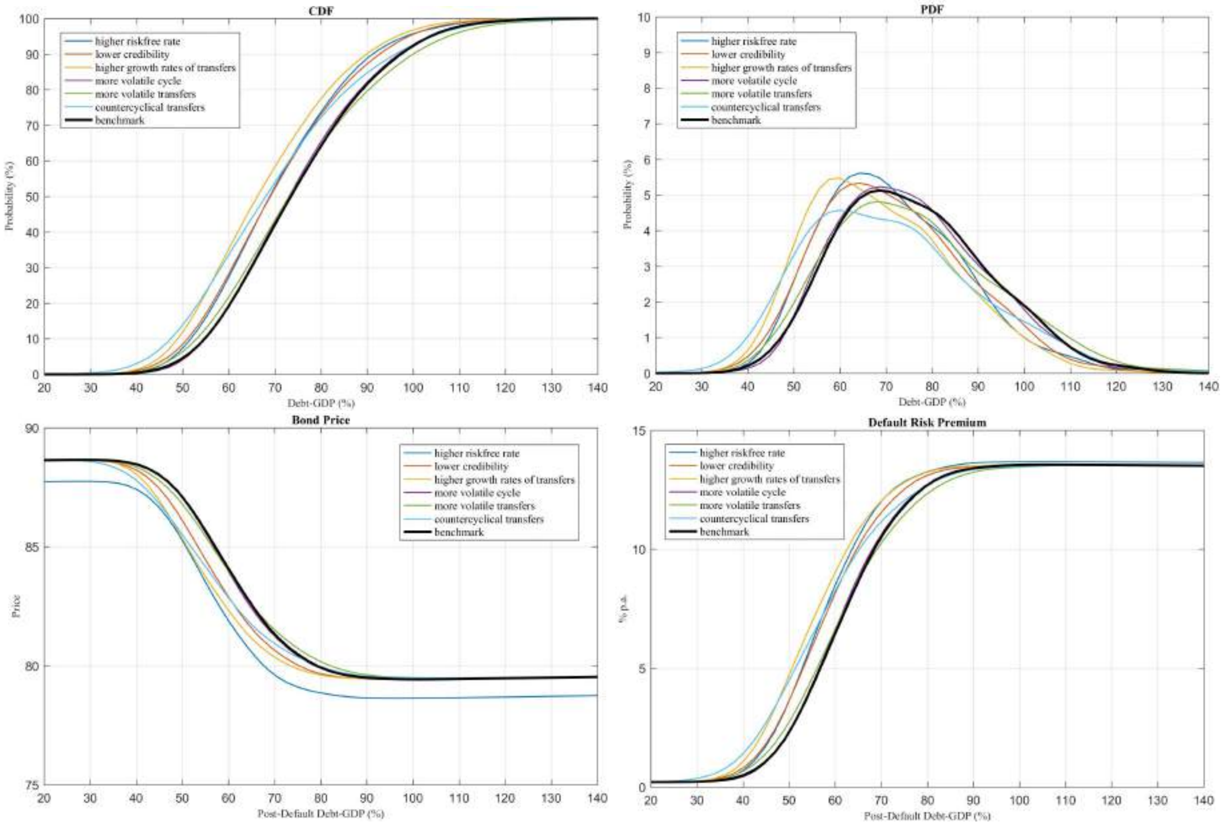
Probability and cumulative distribution functions of the fiscal limit and the corresponding default risk premium estimated for various scenarios. We assume normal times, average initial level of transfers following NPC scenario.

Figure D.10 : Procedure convergence: Fiscal limit and risk premium



Probability and cumulative distribution functions of the fiscal limit and the corresponding default risk premium estimated for various scenarios. We assume normal times, average initial level of transfers following NPC scenario.

Figure D.11 : Unconditional distribution of fiscal limits



Probability and cumulative distribution functions of the fiscal limit and the corresponding default risk premium and bond price estimated for models with regime-switching countercyclical transfers (red lines). In the benchmark model (blue lines) transfers are not cycle-sensitive.

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